ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1 Li_2(-x^2)\tan^{-1}(x)dx$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=\int_0^1 Li_2(-x^2)\tan^{-1}(x)dx=$$

$$\underbrace{=}_{IBP} Li_2(-1)\left(\frac{\pi}{4}-\frac{ln2}{2}\right)+2\int_0^1 \ln(1+x^2)\tan^{-1}(x)dx-\int_0^1\frac{(\ln(1+x^2))^2}{x}dx$$

$$A=\int_0^1 \ln(1+x^2)\tan^{-1}(x)dx \quad \& \quad B=\int_0^1\frac{(\ln(1+x^2))^2}{x}dx$$

$$A=\int_0^1 \ln(1+x^2)\tan^{-1}(x)dx\underbrace{=}_{IBP}\frac{\pi}{4}ln2-\frac{(ln2)^2}{2}-\int_0^1\frac{2x^2\tan^{-1}(x)-xln(1+x^2)}{1+x^2}dx$$

$$\frac{\pi}{4}ln2-\frac{(ln2)^2}{2}-2\int_0^1\tan^{-1}x\,dx+2\int_0^1\frac{\tan^{-1}x}{1+x^2}dx+\int_0^1\frac{xln(1+x^2)}{1+x^2}dx$$

$$A=\left(\frac{\pi}{4}+1\right)ln2-\frac{(ln2)^2}{4}+\frac{\pi^2}{16}-\frac{\pi}{2}$$

$$B=\int_0^1\frac{(\ln(1+x^2))^2}{x}dx\underbrace{=}_{x^2\to x}\int_0^1\frac{(\ln(1+x))^2}{2x}dx\underbrace{=}_{1+x\to x}\int_1^2\frac{(lnx)^2}{2(x-1)}dx$$

$$\underbrace{=}_{x\to\frac{1}{x}}\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2x(1-x)}dx=\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2x(1-x)}dx=\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2x}dx+\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2(1-x)}dx=$$

$$\frac{(ln2)^3}{6}+\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}\int_{\frac{1}{2}}^1\frac{x^{a+n}}{2}dx=\frac{(ln2)^3}{6}+\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}\frac{1-\left(\frac{1}{2}\right)^{a+n+1}}{2(a+n+1)}=$$

$$B=\frac{(ln2)^3}{6}+\sum_{n=1}^{\infty}\frac{1}{(n)^3}-\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{(n)^3}-\frac{(ln2)^2}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n}-ln2\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n^2}$$

$$B=\frac{(ln2)^3}{6}+\zeta(3)-Li_3\left(\frac{1}{2}\right)-\frac{(ln2)^3}{2}-ln2Li_2\left(\frac{1}{2}\right)\Rightarrow B=\frac{\zeta(3)}{8}$$

$$\{\text{note: } Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{(\ln2)^2}{2}\ \&\ Li_3\left(\frac{1}{2}\right)=\frac{(\ln2)^3}{6}-\frac{\pi^2}{12}\ln2+\frac{7\zeta(3)}{8}\ \&\ Li_2(-1)=-\frac{\pi^2}{12}$$

$$ANSWER=-\frac{\pi^3}{48}+\frac{\pi^2 ln2}{24}+2A-B$$

$$=-\frac{\pi^3}{48}+\frac{\pi^2 ln2}{24}+\left(\frac{\pi}{2}+2\right)ln2-\frac{(ln2)^2}{2}+\frac{\pi^2}{8}-\pi-\frac{\zeta(3)}{8}$$