ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\int_0^1 \frac{Li_2(-x)\ln(1+x)}{x(x+1)}dx$$

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$$\Omega=A-B=\int_0^1 \frac{Li_2(-x)\ln(1+x)}{x}dx-\int_0^1 \frac{Li_2(-x)\ln(1+x)}{(x+1)}dx$$

$$A=\int_0^1 \frac{Li_2(-x)\ln(1+x)}{x}dx=-\frac{(Li_2(-1))^2}{2}=\{note: dLi_2(-x)=-\frac{\ln(1+x)}{x}dx$$

$$B=\int_0^1 \frac{Li_2(-x)\ln(1+x)}{(x+1)}dx\underbrace{=}_{IBP}\frac{Li_2(-1)(\ln(2))^2}{2}+\int_0^1 \frac{(\ln(1+x))^3}{2x}dx$$

$$C=\int_0^1 \frac{(\ln(1+x))^3}{2x}dx\underbrace{=}_{1+x\to x}\int_1^2 \frac{(\ln(x))^3}{2(x-1)}dx\underbrace{=}_{x\to\frac{1}{x}}\int_{1/2}^1 \frac{(\ln(x))^3}{2(x-x^2)}dx$$

$$=\int_{1/2}^1 \frac{(\ln(x))^3}{2x}dx+\int_{1/2}^1 \frac{(\ln(x))^3}{2(1-x)}dx=\frac{(\ln(2))^4}{8}+\sum_{n=0}^{\infty}\frac{1}{2}\int_{\frac{1}{2}}^1 x^n(\ln(x))^3\,dx=$$

$$\to D=\sum_{n=0}^{\infty}\frac{1}{2}\int_{\frac{1}{2}}^1 x^n(\ln(x))^3\,dx=\sum_{n=0}^{\infty}\frac{1}{2}\left(\int_0^1 x^n(\ln(x))^3dx-\int_0^{\frac{1}{2}} x^n(\ln(x))^3dx\right)=\Psi-\Phi$$

$$\{note: \int_0^1 x^m(\ln(x))^n\,dx=\frac{(-1)^n n!}{(m+1)^{n+1}}\ \ \{n>-1\wedge m\neq -1\}\}$$

$$\Psi=\sum_{n=0}^{\infty}\frac{1}{2}\int_0^1 x^n(\ln(x))^3dx=\sum_{n=0}^{\infty}\frac{3}{(n+1)^4}=3\zeta(4)$$

$$\Phi=\sum_{n=0}^{\infty}\frac{1}{2}\int_0^{\frac{1}{2}} x^n(\ln(x))^3dx=$$

$$\underbrace{=}_{IBP}-\frac{(\ln(2))^3}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{n+1}-\frac{3}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)}\int_0^{\frac{1}{2}} x^n\,(\ln(x))^2dx$$

$$\underbrace{=}_{IBP}-\frac{(\ln(2))^3}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{n+1}-\frac{3(ln2)^2}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^2}+\sum_{n=0}^{\infty}\frac{3}{(n+1)^2}\int_0^{\frac{1}{2}} x^n\ln(x)dx$$

$$\underbrace{=}_{IBP} -3ln2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^3}-\sum_{n=0}^{\infty}\frac{3}{(n+1)^3}\int_0^{\frac{1}{2}}x^n\,dx=$$

$$\Phi=-\frac{(\ln(2))^3}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{n+1}-\frac{3(ln2)^2}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^2}-3ln2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^3}-3\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^4}$$

$$\Phi=-\frac{(ln2)^3}{2}Li_1\left(\frac{1}{2}\right)-\frac{3(ln2)^2}{2}Li_2\left(\frac{1}{2}\right)-3ln2Li_3\left(\frac{1}{2}\right)-3Li_4\left(\frac{1}{2}\right)$$

$$\Phi=\frac{-(ln2)^4}{4}+\frac{\pi^2(ln2)^2}{8}-\frac{21}{8}ln2\,\zeta(3)-3Li_4\left(\frac{1}{2}\right)$$

$$C=-\frac{(\ln(2))^4}{8}+\Psi-\Phi=3\zeta(4)+\frac{(ln2)^4}{8}+\frac{\pi^2(ln2)^2}{8}+\frac{21}{8}ln2\,\zeta(3)+3Li_4\left(\frac{1}{2}\right)$$

$$\begin{cases}B=\dfrac{Li_2(-1)(\ln(2))^2}{2}+C=-\dfrac{\pi^2(ln2)^2}{24}+C\\ A=-\dfrac{\pi^4}{288}\end{cases}$$

ANSWER: $A-B=\frac{21}{8}ln2\,\zeta(3)+3Li_4\left(\frac{1}{2}\right)+\frac{(ln2)^4}{8}-\frac{\pi^2(ln2)^2}{12}-\frac{53\pi^4}{1440}$

$\{note\ section\}$

$$\zeta(4)=\frac{\pi^4}{90}\ .\ Li_3\left(\frac{1}{2}\right)=\frac{(ln2)^3}{6}-\frac{\pi^2}{12}ln2+\frac{7}{8}\zeta(3)\ .\ Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{(ln2)^2}{2}$$