ROMANIAN MATHEMATICAL MAGAZINE

Find the general form of the integral and if $a_1a_2a_3\dots a_k=1$, $2k>2n+1$ ***prove that:***

$$\xi=\int_0^\infty\frac{x^{2n}}{(x^2+a_1^2)(x^2+a_2^2)(x^2+a_3^2)\dots(x^2+a_k^2)}dx\le$$

$$\le\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{\frac{4n-k}{2}}}{\Pi_{t\in[1;k]\backslash m}(a_t-a_m)}$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by proposer

$$let\ f(z)=\frac{z^{2n}}{(z^2+a_1^2)(z^2+a_2^2)(z^2+a_3^2)\dots(z^2+a_k^2)}=\frac{z^{2n}}{\Pi_{i=1}^k(z^2+a_i^2)}$$

$$Consider:\ \Gamma\ U(-R;R)\ is\ the\ anti-clockwise\ and\ semi-circular\ contour\ in\ the\ upper\ half\ of\ C\ plane$$

$$\oint_C f(z)\,dz=\int_\Gamma f(z)\,dz+\int_{-R}^R f(x)\,dx$$

$$\oint_C f(z)\,dz=2\pi i\sum Res(f;poles\ of\ f)=2\pi i\sum_{m=1}^k\frac{(a_m i)^{2n}}{2a_m i\ \Pi_{t\in[1;k]\,\backslash\,m}(a_t^2-a_m^2)}=$$

$$=\pi\sum_{m=1}^k\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\,\backslash\,m}(a_t^2-a_m^2)}$$

$$f(-x)=f(x)=>\int_{-R}^R f(x)\,dx=2\int_0^R f(x)\,dx=2\xi$$

$$z=Re^{i\theta},dz=izd\theta\,,\theta[0;\pi]=>\int_\Gamma f(z)\,dz=\int_0^\pi\frac{(Re^{i\theta})^{2n}}{\Pi_{i=1}^k(R^2e^{2i\theta}+a_i^2)}\times iRe^{i\theta}d\theta$$

$$=\int_0^\pi\frac{iR^{2n+1}e^{2in\theta+i\theta}}{\Pi_{i=1}^k(R^2e^{2i\theta}+a_i^2)}d\theta$$

$$\left|\int_\Gamma f(z)\,dz\right|\le\int_\Gamma|f(z)|\,dz=\int_0^\pi\frac{|i||R^{2n+1}||e^{i\theta(2n+1)}|}{\Pi_{i=1}^k|R^2e^{2i\theta}+a_i^2|}d\theta\le\int_0^\pi\frac{R^{2n+1}}{\Pi_{i=1}^k(R^2)}d\theta=\frac{R^{2n+1}}{R^{2k}}\pi=0$$

$$2\xi=\pi\sum_{m=1}^k\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\backslash m}(a_t^2-a_m^2)}$$

$$\xi=\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}(a_t-a_m)(a_t+a_m)}\le\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}\left(2\sqrt{a_t a_m}(a_t-a_m)\right)}$$

$$=\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{2^{k-1}a_m^{\frac{k-1}{2}}\times\sqrt{\frac{\Pi_{j=1}^{k}(a_j)}{a_m}}\;\Pi_{t\in[1;k]\setminus m}(a_t-a_m)}=$$

$$=\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{a_m^{\frac{k}{2}-1}\;\Pi_{t\in[1;k]\setminus m}(a_t-a_m)}=\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{\frac{4n-k}{2}}}{\Pi_{t\in[1;k]\setminus m}(a_t-a_m)}$$

$$\int_0^\infty\frac{x^{2n}}{(x^2+a_1^2)(x^2+a_2^2)(x^2+a_3^2)\dots(x^2+a_k^2)}dx=\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}(a_t^2-a_m^2)}$$

$$\int_0^\infty\frac{x^{2n}}{(x^2+a_1^2)(x^2+a_2^2)(x^2+a_3^2)\dots(x^2+a_k^2)}dx\le\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{\frac{4n-k}{2}}}{\Pi_{t\in[1;k]\setminus m}(a_t-a_m)}$$