ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^\infty\int_0^\infty \frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{x^8y^2+x^8y+y^2+y+x^4y^2+x^4y}dxdy=-\frac{\pi^2\zeta(3)}{36}(3+2\sqrt{3})$$

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$$I=\int_0^\infty\int_0^\infty \frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{x^8y^2+x^8y+y^2+y+x^4y^2+x^4y}dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{(y+y^2)(x^8+x^4+1)}dxdy=$$

$$=2\int_0^\infty \frac{(1+x^4)\ln(x)}{x^8+x^4+1}dx\int_0^\infty \frac{\ln(y)\ln(1+y)}{y(1+y)}dy=2JK$$

$$J=\int_0^\infty \frac{(1+x^4)\ln(x)}{x^8+x^4+1}dx=\int_0^\infty \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=$$

$$=\int_0^1 \frac{(1-x^8)\ln(x)}{1-x^{12}}dx+\int_1^\infty \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=J_1+J_2$$

$$J_1=\int_0^1 \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=\sum_{n=0}^\infty\int_0^1 (x^{12n}-x^{8+12n})\ln(x)\,dx=$$

$$=\sum_{n=0}^\infty \frac{1}{(12n+9)^2}-\sum_{n=0}^\infty \frac{1}{(12n+1)^2}=\frac{1}{144}\left(\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{1}{12}\right)\right)$$

$$J_2=\int_1^\infty \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=\int_0^1 \frac{\left(1-\frac{1}{x^8}\right)\ln(x)}{1-\frac{1}{x^{12}}}\frac{dx}{x^2}=\int_0^1 \frac{(x^{10}-x^2)\ln(x)}{1-x^{12}}dx=$$

$$=\sum_{n=0}^\infty\int_0^1 (x^{12n+10}-x^{12n+2})\ln(x)dx=\sum_{n=0}^\infty \frac{1}{(12n+3)^2}-\sum_{n=0}^\infty \frac{1}{(12n+11)^2}=$$

$$=\frac{1}{144}\left(\psi^{(1)}\left(\frac{1}{4}\right)-\psi^{(1)}\left(\frac{11}{12}\right)\right)$$

$$J=J_1+J_2=\frac{1}{144}\left(\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{1}{12}\right)+\psi^{(1)}\left(\frac{1}{4}\right)-\psi^{(1)}\left(\frac{11}{12}\right)\right)$$

Notes:

polygamma reflection formula:

$$\left\{(-1)^m\psi^{(m)}(1-x)-\psi^{(m)}(x)=\pi\frac{d}{dx^m}\cot(\pi x)\right\}$$

$$J=J_1+J_2=\frac{1}{144}\left(\psi^{(1)}\left(1-\frac{1}{4}\right)+\psi^{(1)}\left(\frac{1}{4}\right)-\psi^{(1)}\left(1-\frac{11}{12}\right)-\psi^{(1)}\left(\frac{11}{12}\right)\right)=$$

$$=\frac{1}{144}\left(2\pi^2-4\pi^2(2+\sqrt{3})\right)=-\frac{\pi^2}{72}(3+2\sqrt{3})$$

$$K=\int_0^\infty \frac{\ln(y)\ln(1+y)}{y(1+y)}dy=\int_0^1 \frac{\ln(y)\ln(1+y)}{y(1+y)}dy+\int_1^\infty \frac{\ln(y)\ln(1+y)}{y(1+y)}dy=K_1+K_2$$

$$K_1=\int_0^1 \frac{\ln(y)\ln(1+y)}{y(1+y)}dday=\int_0^1 \frac{\ln(y)\ln(1+y)}{y}dy-\int_0^1 \frac{\ln(y)\ln(1+y)}{1+y}dy=$$

$$=-\sum_{n=1}^\infty \frac{(-1)^n}{n}\int_0^1 y^{n-1}\ln(y)dy-[\frac{\ln(y)\ln^2(1+y)}{2}]_0^1+\frac{1}{2}\int_0^1 \frac{\ln^2(1+y)}{y}dy=$$

$$=\sum_{n=1}^\infty \frac{(-1)^n}{n^3}+\frac{1}{2}\int_0^1 \frac{\ln^2(1+y)}{y}dy,\left\{\frac{1}{1+y}=t,dt=-t^2dy,t\left[\frac{1}{2};1\right]\right\}$$

$$=-\eta(3)+\frac{1}{2}\int_{\frac{1}{2}}^1 \frac{\ln^2(t)}{\frac{1}{t}-1}\frac{dt}{t^2}=-\frac{3}{4}\zeta(3)+\frac{1}{2}\int_{\frac{1}{2}}^1 \frac{\ln^2(t)}{t(1-t)}dt=$$

$$=-\frac{3}{4}\zeta(3)+\frac{1}{2}\sum_{n=0}^\infty \int_{\frac{1}{2}}^1 t^{n-1}\ln^2(t)dt=-\frac{3}{4}\zeta(3)+\frac{\zeta(3)}{8}=-\frac{5\zeta(3)}{8}$$

$$K_2=\int_1^\infty \frac{\ln(y)\ln(1+y)}{y(1+y)}dy=\int_0^1 \frac{\ln\left(\frac{1}{y}\right)\ln\left(1+\frac{1}{y}\right)}{y\left(1+\frac{1}{y}\right)}dy=\int_0^1 \frac{\ln^2(y)-\ln(y)\ln(1+y)}{1+y}dy=$$

$$=\sum_{n=0}^\infty (-1)^n\int_0^1 y^n\ln^2(y)dy-[\frac{\ln(y)\ln^2(1+y)}{2}]_0^1+\frac{1}{2}\int_0^1 \frac{\ln^2(1+y)}{y}dy=2\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^3}$$

$$+\frac{\zeta(3)}{8}=2\eta(3)+\frac{\zeta(3)}{8}=\frac{13\zeta(3)}{8}$$

$$K=K_1+K_2=\frac{13\zeta(3)}{8}-\frac{5\zeta(3)}{8}=\zeta(3)$$

$$\int_0^\infty\int_0^\infty \frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{x^8y^2+x^8y+y+y^2+x^4y^2+x^4y}dxdy=2JK=-\frac{\pi^2\zeta(3)\left(3+2\sqrt{3}\right)}{36}$$