Find a closed form:

$$\Omega=\int_0^1\int_0^1\left(\frac{\ln(1+xy)}{1+xy}\right)^2 dxdy$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1\left(\frac{\ln(1+xy)}{1+xy}\right)^2 dxdy,\qquad \{\int_0^1\int_0^1 f(xy)dxdy=-\int_0^1\ln(x)f(x)\,dx\}$$

$$\Omega=-\int_0^1\frac{\ln(x)\ln^2(1+x)}{(1+x)^2}dx, using\ \dot{I}BP\ method$$

$$\begin{Bmatrix} u=\ln(x)\ln^2(1+x),\ \ du=\left(\frac{2\ln(x)\ln(1+x)}{1+x}+\frac{\ln^2(1+x)}{x}\right)dx \\ v=\int\frac{1}{(1+x)^2}dx=-\frac{1}{1+x}\end{Bmatrix}$$

$$\Omega=[\frac{\ln(x)\ln^2(1+x)}{1+x}]_0^1-2\int_0^1\frac{\ln(x)\ln(1+x)}{(1+x)^2}dx-\int_0^1\frac{\ln^2(1+x)}{x(1+x)}dx$$

$$\Omega=2\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\frac{\ln(x)\ln(1+x)}{(1+x)^2}dx\ ,using\ IBP\ method$$

$$\begin{Bmatrix} u=\ln(x)\ln(1+x),\ \ du=\left(\frac{\ln(x)}{1+x}+\frac{\ln(1+x)}{x}\right)dx \\ v=\int\frac{1}{(1+x)^2}dx=-\frac{1}{1+x}\end{Bmatrix}$$

$$\Omega_1=[-\frac{\ln(x)\ln(1+x)}{1+x}]_0^1+\int_0^1\frac{\ln(x)}{(1+x)^2}dx+\int_0^1\frac{\ln(1+x)}{x}dx-\int_0^1\frac{\ln(1+x)}{1+x}dx=$$

$$=-\sum_{n=1}^{\infty}n(-1)^n\int_0^1 x^{n-1}\ln(x)\,dx-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}dx-[\frac{\ln^2(1+x)}{2}]_0^1=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\frac{\ln^2(2)}{2}=-\ln(2)+\frac{\zeta(2)}{2}-\frac{\ln^2(2)}{2}$$

$$\Omega_2=\int_0^1\frac{\ln^2(1+x)}{x(1+x)}dx,\quad using\ IBP\ method$$

$$\begin{Bmatrix} u=\ln^2(1+x), \quad du=\frac{2\ln(1+x)}{1+x}dx \\ v=\int\frac{1}{x(1+x)}dx=\int\frac{1}{x}dx-\int\frac{1}{1+x}dx=\ln(x)-\ln(1+x) \end{Bmatrix}$$

$$\Omega_2=[\ln^2(1+x)(\ln(x)-\ln(1+x))]_0^1-2\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}dx+2\int_0^1\frac{\ln^2(1+x)}{1+x}dx =$$

$$=-\ln^3(2)-2I+2J$$

$$I=\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}dx=-\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1x^n\ln(x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(n+1)^2}=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{n+1}-\frac{1}{n+1}\right)}{(n+1)^2}=\sum_{n=1}^{\infty}\frac{(-1)^nH_{n+1}}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^3}=$$

$$=1-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}-\frac{3\zeta(3)}{4}-1=\frac{5\zeta(3)}{8}-\frac{3\zeta(3)}{4}=-\frac{\zeta(3)}{8}$$

$$J=\int_0^1\frac{\ln^2(1+x)}{1+x}dx=\int_1^2\frac{\ln^2(x)}{x}dx=[\ln^3(x)]_1^2-2\int_1^2\frac{\ln^2(x)}{x}dx=\ln^3(2)-2J$$

$$3J=\ln^3(2)\quad J=\frac{\ln^3(2)}{3}$$

$$\Omega_2=-\ln^3(2)+\frac{\zeta(3)}{4}+\frac{2\ln^3(2)}{3}=\frac{\zeta(3)}{4}-\frac{\ln^3(2)}{3}$$

$$\int_0^1\int_0^1\left(\frac{\ln(1+xy)}{1+xy}\right)^2dxdy=-2\Omega_1-\Omega_2=2\ln(2)-\frac{\pi^2}{6}+\ln^2(2)-\frac{\zeta(3)}{4}+\frac{\ln^3(2)}{3}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\mathrm{I}=\int_0^1\int_0^1\frac{\log^2(1+xy)}{(1+xy)^2}dx\,\mathrm{dy}\Bigg]_{xy=m}=\int_0^1\frac{1}{x}\int_0^1\frac{\log^2(1+m)}{(1+m)^2}dm\,\mathrm{dx}\overset{IBP}{=}$$

$$\overset{IBP}{=}\log(x)\int_0^x\frac{\log^2(1+m)}{(1+m)^2}dm\Bigg]_{x=0}^{x=1}-\int_0^1\frac{\log(x)\log^2(1+x)}{(1+x)^2}dx$$

$$=-\int_0^1\frac{\log(x)\log^2(1+x)}{(1+x)^2}dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\frac{\log(x)\log^2(1+x)}{(1+x)^2}\Bigg]_{x=0}^{x=1}-\int_0^1\frac{\log^2(1+x)}{x(1+x)}dx-2\int_0^1\frac{\log(x)\log(1+x)}{(1+x)^2}dx=$$

$$= -\int_0^1 \frac{\log^2(1+x)}{x(1+x)} dx - 2\int_0^1 \frac{\log(x)\log(1+x)}{(1+x)^2} dx = -I_1 - 2I_2$$

$$I_1 = \int_0^1 \frac{\log^2(1+x)}{x(1+x)} dx = \int_0^1 \frac{\log^2(1+x)}{x} dx - \int_0^1 \frac{\log^2(1+x)}{1+x} dx = \frac{1}{4}\varsigma(3) - \frac{1}{3}\log^3(2)$$

$$I_2 = \int_0^1 \frac{\log(x)\log(1+x)}{(1+x)^2} dx \overset{IBP}{=} -\frac{\log(x)\log(1+x)}{1+x}\Bigg]_{x=0}^{x=1} + \int_0^1 \frac{\log(1+x)}{x(1+x)} dx + \int_0^1 \frac{\log(x)}{(1+x)^2} dx$$

$$= \int_0^1 \frac{\log(1+x)}{x} dx - \int_0^1 \frac{\log(1+x)}{1+x} dx + \int_0^1 \frac{\log(x)}{(1+x)^2} dx = \frac{1}{2}\varsigma(2) - \frac{1}{2}\log^2(2) - \log(2)$$

$$I = -I_1 - 2I_2 = -\frac{1}{4}\varsigma(3) - \varsigma(2) + \frac{1}{3}\log^3(2) + \log^2(2) + 2\log(2)$$