ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\ln(1+x+y+xy)\tan^{-1}(1+x)}{(1+x)(1+y)}dxdy$$

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$$\Omega=\int_0^1\int_0^1\frac{\ln\big((1+x)(1+y)\big)\tan^{-1}(1+x)}{(1+x)(1+y)}dxdy=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\int_0^1\frac{\ln(1+x)\tan^{-1}(1+x)}{(1+x)(1+y)}dxdy=\ln(1+y)\,|_0^1\int_0^1\frac{\ln(1+x)tan^{-1}(1+x)}{1+x}dx=$$

$$=\ln(2)\bullet I$$

$$I=\int_0^1\frac{\ln(1+x)\tan^{-1}(1+x)}{1+x}dx=\int_1^2\frac{\ln(x)\tan^{-1}(x)}{x}dx=$$

$$=\int_0^2\frac{\ln(x)\tan^{-1}(x)}{x}dx-\int_0^1\frac{\ln(x)\tan^{-1}(x)}{x}dx=I_1-I_2$$

$$I_1=\int_0^2\frac{\ln(x)\tan^{-1}(x)}{x}dx,\qquad using\ \dot{I}BP\ method$$

$$\left\{\begin{matrix} u=\ln(x),\qquad du=\dfrac{dx}{x}\\ v=\displaystyle\int\frac{\tan^{-1}(x)}{x}dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int x^{2n}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)^2}\end{matrix}\right\}\dot{I}BP$$

$$note:\left\{\sum_{n=0}^{\infty}a_{2n+1}=\frac{1}{2}\left(\sum_{n=1}^{\infty}a_n-\sum_{n=0}^{\infty}(-1)^n a_n\right)\right\}$$

$$v=\frac{1}{2i}\sum_{n=1}^{\infty}\frac{i^n x^n}{n^2}-\frac{1}{2i}\sum_{n=1}^{\infty}\frac{(-1)^n i^n x^n}{n^2}=\frac{i}{2}\big(Li_2(-ix)-Li_2(ix)\big)$$

$$I_1=[\frac{i\ln(x)}{2}(Li_2(-ix)-Li_2(ix)]_0^2-\frac{i}{2}\int_0^2\frac{Li_2(-ix)-Li_2(ix)}{x}dx=$$

$$=\frac{i\ln(2)}{2}\big(Li_2(-2i)-Li_2(2i)\big)-\frac{i}{2}[Li_3(-ix)-Li_3(ix)]_0^2=$$

$$=\frac{i}{2}\ln(2)\big(Li_2(-2i)-Li_2(2i)\big)-\frac{i}{2}\big(Li_3(-2i)-Li_3(2i)\big)$$

$$I_2=\int_0^1\frac{\ln(x)\tan^{-1}(x)}{x}dz=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 x^{2n}\ln(x)\,dx=$$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = -\beta(3) = -\frac{\pi^3}{32}$$

$$\Omega_1 = \ln(2)\,(I_1 - I_2) =$$

$$= \frac{i\ln^2(2)}{2}\big(Li_2(-2i) - Li_2(2i)\big) - \frac{i\ln(2)}{2}\big(Li_3(-2i) - Li_3(2i)\big) + \frac{\pi^3}{32}\ln(2)$$

$$\Omega_2 = \int_0^1\int_0^1 \frac{\ln(1+y)\tan^{-1}(1+x)}{(1+y)(1+x)}dx =$$

$$= \left[\frac{\ln^2(1+y)}{2}\right][0;1]\int_0^1 \frac{\tan^{-1}(1+x)}{1+x}dx = \frac{\ln^2(2)}{2}\int_1^2 \frac{\tan^{-1}(x)}{x}dx =$$

$$= \frac{\ln^2(2)}{2}\left(\int_0^2 \frac{\arctan(x)}{x}dx - \int_0^1 \frac{\arctan(x)}{x}dx\right) =$$

$$= \frac{i\ln^2(2)}{4}\big(Li_2(-2i) - Li_2(2i)\big) - \frac{\ln^2(2)}{2}\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2} =$$

$$= i\frac{\ln^2(2)}{4}\big(\mathrm{Li}_2(-2\mathrm{i}) - \mathrm{Li}_2(2\mathrm{i})\big) - \frac{\ln^2(2)}{2}\mathrm{G}$$

$$\int_0^1\int_0^1 \frac{\ln(1+x+y+xy)\arctan(1+\mathrm{x})}{(1+\mathrm{x})(1+y)}dxdy = \Omega_1 + \Omega_2 =$$

$$= 3i\frac{\ln^2(2)}{2}\big(Li_2(-2i) - Li_2(2i)\big) - \frac{i\ln(2)}{2}\big(Li_3(-2i) - Li_3(2i)\big) + \frac{\pi^3}{32}\ln(2) - \frac{\ln^2(2)}{2}\mathrm{G}$$