ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1 Li_2(-x^2)\arctan(x)\,dx$$

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Solution by proposer

$$Li_2(-x^2)=-\int_0^1\frac{\ln(1+x^2y)}{y}dy\ \ =>\ \Omega=-\int_0^1\int_0^1\frac{\arctan(x)\ln(1+x^2y)}{y}\mathrm{dxdy}$$

$$\begin{cases} u=\ln(1+x^2y) & du=\dfrac{2xy}{1+x^2y}\\ dv=\arctan(x) & v=\arctan(x)-\dfrac{1}{2}\ln(1+x^2)\end{cases}$$

$$\Omega=-\int_0^1\frac{1}{\mathrm{y}}\left[\ln(1+y)\left(\frac{\pi}{4}-\frac{1}{2}\ln(2)\right)-\int_0^1\frac{2x^2y\arctan(x)}{1+x^2y}dx+\int_0^1\frac{xy\ln(1+x^2)}{1+x^2y}dx\right]dy$$

$$\Omega=-\frac{\pi}{4}\int_0^1\frac{\ln(1+y)}{y}dy$$

$$+\frac{\ln(2)}{2}\int_0^1\frac{\ln(1+y)}{y}dy+2\int_0^1\int_0^1\frac{x^2\arctan(x)}{1+x^2y}dxdy-\int_0^1\int_0^1\frac{x\ln(1+x^2)}{1+x^2y}dxdy=$$

$$=-\frac{\pi^3}{48}+\frac{\pi^2\ln(2)}{24}+2\int_0^1\ln(1+x^2)\arctan(x)\,dx\ -\int_0^1\frac{\ln^2(1+x^2)}{x}dx$$

$$=-\frac{\pi^3}{48}+\frac{\pi^2\ln(2)}{24}-\frac{\zeta(3)}{8}+\Omega_1$$

$$\Omega_1=2\int_0^1\ln(1+x^2)\arctan(x)\,dx$$

$$=2[\arctan(x)\,(x\ln(1+x^2)+2\arctan(x)-2x)]_0^1$$

$$-2\int_0^1\frac{xln(1+x^2)+2\arctan(x)-2x}{1+x^2}dx$$

$$\Omega_1=\frac{\pi}{2}(\ln(2)+\frac{\pi}{2}-2)-2\int_0^1\frac{\mathrm{x}\ln(1+\mathrm{x}^2)}{1+x^2}dx-4\int_0^1\frac{\arctan(x)}{1+x^2}dx+4\int_0^1\frac{x}{1+x^2}dx$$

$$\Omega_1=\frac{\pi\ln(2)}{2}+\frac{\pi^2}{4}-\pi-\frac{\ln^2(2)}{2}-\frac{\pi^2}{8}+2\ln(2)$$

$$\Omega=-\frac{\pi^3}{48}+\frac{\pi^2\ln(2)}{24}-\frac{\zeta(3)}{8}+\frac{\pi\ln(2)}{2}+\frac{\pi^2}{4}-\pi-\frac{\ln^2(2)}{2}-\frac{\pi^2}{8}+2\ln(2)$$

$$\Omega=-\frac{\pi^3}{48}+\frac{\pi^2\ln(2)}{24}-\frac{\zeta(3)}{8}+\frac{\pi\ln(2)}{2}-\pi-\frac{\ln^2(2)}{2}+\frac{\pi^2}{8}+2\ln(2)$$