ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}\,dx\,dy$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

$$We\ know,\ \int_0^1\int_0^1 f(xy)dx\,dy=-\int_0^1\ln(t)f(t)\,dt$$

$$I=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dx\,dy=-\int_0^1\frac{\ln^2(x)\ln(1+2x)}{x}dx=$$

$$=^{IBP}\left[\ln^2(x)\int d\left(\mathrm{Li}_2(-2x)\right)\right]_0^1-2\int_0^1\frac{\ln(x)Li_2(-2x)}{x}dx=^{IBP}-2\left[\ln(x)\int d\left(Li_3(-2x)\right)\right]_0^1$$

$$+2\int_0^1\frac{Li_3(-2x)}{x}dx=2\int_0^1 d(Li_4(-2x)dx=2Li_4(-2)$$

$$Li_4(-z)+Li_4\left(-\frac{1}{z}\right)=-\frac{7\pi^4}{360}-\pi^2\ln^2(z)-\frac{\ln^4(z)}{24}$$

$$I=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dx\,dy$$

$$=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)-\frac{7\pi^4}{360}-\frac{\pi^2\ln^2(2)}{12}-\frac{\ln^4(2)}{24}$$

Solution 2 by Arowolo Isaiah-Nigeria

$$I=-\int_0^1\frac{\ln^2(x)\ln(1+2x)}{x}=^{IBP}-\left(\left[\frac{\ln^3(x)\ln(1+2x)}{3}\right]_0^1-\frac{2}{3}\int_0^1\frac{\ln^3(x)}{1+2x}\mathrm{dx}\right)$$

$$I=\frac{2}{3}\int_0^1\frac{\ln^3(x)}{x}dx=\frac{2}{3}(\sum_{n=0}^{\infty}(-2)^n\int_0^1 x^{3n}\ln^3(x)\,\mathrm{dx})$$

$$=-\frac{1}{3}(\sum_{n=1}^{\infty}(-2)^n\int_0^1 x^{n-1}\ln^3(x)\,dx)$$

$$I=-\frac{1}{3}\sum_{n=1}^{\infty}(-2)^n\left(\frac{d^3}{dn^3}\left(\frac{1}{n}\right)\right)=2\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}=2Li_4(-2)=Li_4(-2)+Li_4(-2)$$

$$I=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)+\ln^2(2)\mathrm{Li}_2(-1)+2\mathrm{Li}_4(-1)-\frac{\ln^4(2)}{24}$$

$$I=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)+\ln^2(2)\left(-\frac{1}{2}\zeta(2)\right)+2\left(-\frac{7}{8}\zeta(4)\right)-\frac{\ln^4(2)}{24}$$

$$I=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)-\frac{\pi^2\ln^2(2)}{12}-\frac{7\pi^4}{360}-\frac{\ln^4(2)}{24}$$