ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \int_0^1 \frac{\ln(1+x^2)(\tan^{-1}(x)+x)}{1+x^2}dx$$

Proposed by Abbaszade Yusif, Shirvan Tahirov-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega = \int_0^1 \frac{\ln(1+x^2)\tan^{-1}x}{1+x^2}dx + \int_0^1 \frac{\ln(1+x^2)\,x}{1+x^2}dx = A + B$$

$$A = \int_0^1 \frac{\ln(1+x^2)\tan^{-1}x}{1+x^2}dx$$

$$\underbrace{=}_{\tan^{-1}(x)\to x} -2\int_0^{\frac{\pi}{4}} x\ln\cos x\,dx = 2\int_0^{\frac{\pi}{4}} x\ln 2dx + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} x\cos(2nx)dx$$

$$= \frac{\pi^2}{16}\ln 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} x\cos(2nx)dx = \{\text{IBP METHOD}\}$$

$$\frac{\pi^2}{16}\ln 2 + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\frac{n\pi}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^{\frac{\pi}{4}}\sin(2nx)\,dx =$$

$$\frac{\pi^2}{16}\ln 2 + \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}\cos\frac{n\pi}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}$$

$$A = \frac{\pi^2}{16}\ln 2 + \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2} - \frac{7}{16}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n)^3} = \frac{\pi^2}{16}\ln 2 - \frac{\pi}{4}C + \frac{21}{64}\zeta(3)$$

$$B = \int_0^1 \frac{\ln(1+x^2)\,x}{1+x^2}dx = \frac{(\ln(1+x^2))^2}{4}\Big]_0^1 = \frac{(\ln 2)^2}{4}$$

$$\text{ANSWER} = \text{A} + \text{B} = \frac{\pi^2}{16}\ln 2 - \frac{\pi}{4}\text{C} + \frac{21}{64}\zeta(3) + \frac{(\ln 2)^2}{4}$$