ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dxdy$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dxdy=$$

$$\underbrace{=}_{xy\to t}\int_0^1\int_0^y\frac{\ln(t)\ln(1+2t)}{ty}dtdy=-\sum_{n=1}^{\infty}\frac{(-2)^n}{n}\int_0^1\int_0^y\frac{1}{y}\ln(t)\,t^{n-1}dtdy=\{IBP\ method\}$$

$$=-\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\ln(y)\,y^{n-1}dy+\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\int_0^y\frac{t^{n-1}}{y}dtdy=A+B$$

$$A=-\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\ln(y)\,y^{n-1}dy=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}\quad\left\{note\int_0^1x^m\left(lnx\right)^ndx=\frac{(-1)^nn!}{(m+1)^{n+1}}\right\}$$

$$\mathrm{B}=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\int_0^y\frac{t^{n-1}}{y}dtdy=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^3}\int_0^1y^{n-1}dy=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}$$

$$I=A+B=2\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}=2Li_4(-2)\quad\left\{note\ \ Li_4(-2)=-Li_4\left(\frac{1}{2}\right)-\frac{7\pi^4}{360}-\frac{\pi^2(ln2)^2}{12}-\frac{(ln2)^4}{24}\right\}$$

$$\mathrm{I}=Li_4(-2)-Li_4\left(\frac{-1}{2}\right)-\frac{\pi^2(ln2)^2}{12}-\frac{7\pi^4}{360}-\frac{(ln2)^4}{24}$$