ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\ln(1+x+y+xy)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy$$

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$$\Omega=\int_0^1\int_0^1\frac{\ln(1+x+y+xy)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy=$$

$$=\int_0^1\int_0^1\frac{\ln(1+y)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy+\int_0^1\int_0^1\frac{\ln(1+x)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy=\mathrm{A}+\mathrm{B}$$

$$\mathrm{B}=\int_0^1\frac{dy}{1+y}\int_0^1\frac{\ln(1+x)\tan^{-1}(x+1)}{x+1}dx\underbrace{=}_{x+1\to x}\ln(2)\int_1^2\frac{\ln(x)\tan^{-1}(x)}{x}dx\ \ \{\mathrm{IBP}\}$$

$$=\frac{(ln2)^3\tan^{-1}(2)}{2}-\frac{ln2}{2}\int_1^2\frac{(lnx)^2}{1+x^2}dx=\frac{(ln2)^3\tan^{-1}(2)}{2}-\frac{ln2}{2}C$$

$$C=\int_1^2\frac{(lnx)^2}{1+x^2}dx\underbrace{=}_{x\to 1/x}\frac{1}{2}\int_{\frac{1}{2}}^1\frac{(lnx)^2}{1+ix}dx+\frac{1}{2}\int_{\frac{1}{2}}^1\frac{(lnx)^2}{1-ix}dx$$

$$=\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{2}\int_{\frac{1}{2}}^1(lnx)^2x^n\,dx\underbrace{=}_{IBP}$$

$$-\frac{(ln2)^2}{4}\sum_{n=0}^{\infty}\left(\frac{i^n+(-i)^n}{n+1}\right)(2^{-n})-\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{n+1}\int_{\frac{1}{2}}^1x^n\ln(x)\,dx\quad\{\mathrm{IBP}\}$$

$$=-\frac{(ln2)^2}{4}\sum_{n=0}^{\infty}\left(\frac{i^n+(-i)^n}{n+1}\right)(2^{-n})-\frac{ln2}{2}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^2}(2^{-n})+\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^2}\int_{\frac{1}{2}}^1x^ndx$$

$$=$$

$$C=-\frac{(ln2)^2}{4}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{n+1}(2^{-n})-\frac{ln2}{2}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^2}(2^{-n})-\frac{1}{2}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^3}(2^{-n})$$
$$+\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^3}$$

$$B=\frac{(ln2)^3\tan^{-1}(2)}{2}+\frac{(ln2)^3}{8}\sum_{n=1}^{\infty}\frac{\left(\frac{i}{2}\right)^{n-1}+\left(-\frac{i}{2}\right)^{n-1}}{n}+\frac{(ln2)^2}{4}\sum_{n=1}^{\infty}\frac{\left(\frac{i}{2}\right)^{n-1}+\left(-\frac{i}{2}\right)^{n-1}}{n^2}$$

$$+\frac{ln2}{4}\sum_{n=1}^{\infty}\frac{\left(\frac{i}{2}\right)^{n-1}+\left(-\frac{i}{2}\right)^{n-1}}{n^3}-\frac{ln2}{2}\sum_{n=1}^{\infty}\frac{i^{n-1}+(-i)^{n-1}}{n^3}$$

$$B=\frac{(ln2)^3\tan^{-1}(2)}{2}+\frac{(ln2)^3}{2}\tan^{-1}\left(\frac{1}{2}\right)+\frac{(ln2)^2}{2i}\left(Li_2\left(\frac{i}{2}\right)-Li_2\left(\frac{-i}{2}\right)\right)$$

$$-\frac{ln2}{2i}\left(Li_3(i)-Li_3(-i)+Li_3\left(\frac{-i}{2}\right)-Li_3\left(\frac{i}{2}\right)\right)$$

$$\int_0^1\int_0^1\frac{\ln(1+y)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy\underbrace{=}_{1+y\to y}\int_1^2\frac{\ln(y)}{y}dy\int_1^2\frac{\tan^{-1}(x)}{x}dx$$

$$=\frac{(ln2)^2}{2}\int_1^2\frac{\tan^{-1}(x)}{x}dx$$

$$\frac{(ln2)^2}{2}\int_1^2\frac{\tan^{-1}(x)}{x}dx\underbrace{=}_{IBP}\frac{(ln2)^3}{2}\tan^{-1}(2)-\frac{(ln2)^2}{2}\int_1^2\frac{\ln(x)}{x^2+1}dx\underbrace{=}_{x\to\frac{1}{x}}$$

$$\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^2}{2}\int_{\frac{1}{2}}^1\frac{\ln(x)}{x^2+1}\mathrm{dx}=\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^2}{2}W$$

$$W=\int_{\frac{1}{2}}^1\frac{\ln(x)}{x^2+1}\mathrm{dx}=\frac{1}{2}\int_{\frac{1}{2}}^1\frac{\ln(x)}{1+ix}dx+\frac{1}{2}\int_{\frac{1}{2}}^1\frac{\ln(x)}{1-ix}dx)=\sum_{n=0}^{\infty}\frac{(i^n+(-i)^n)}{2}\int_{1/2}^1x^n\ln(x)\,dx$$

$$\{IBP\}\quad=\frac{ln2}{4}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n}-\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{2(n+1)}\int_{1/2}^1x^ndx$$

$$W=\frac{ln2}{4}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n}-\sum_{n=1}^{\infty}\frac{i^{n-1}+(-i)^{n-1}}{2n^2}+\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{4n^2}$$

$$A=\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^3}{8}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n}-\frac{(ln2)^2}{4}\sum_{n=1}^{\infty}\frac{i^{n-1}+(-i)^{n-1}}{n^2}$$

$$+\frac{(ln2)^2}{8}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n^2}$$

$$A=\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^3}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

$$-\frac{(ln2)^2}{4i}\left(Li_2(i)-Li_2(-i)-Li_2\left(\frac{i}{2}\right)+Li_2\left(\frac{-i}{2}\right)\right)$$

$$\text{ANSWER} = A + B = \frac{\pi(ln2)^3}{2} + \frac{3(ln2)^2}{4i}\left(Li_2\left(\frac{i}{2}\right) - Li_2\left(\frac{-i}{2}\right)\right)$$

$$-\frac{ln2}{2i}\left(Li_3(i) - Li_3(-i) + Li_3\left(\frac{-i}{2}\right) - Li_3\left(\frac{i}{2}\right)\right)$$