ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^\infty e^{-x^2}\sqrt{(\cosh^2(x)-1)}\,dx=\frac{e^{\frac{1}{4}}}{2}\sqrt{\pi}\,\text{erf}\left(\frac{1}{2}\right)$$

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$$\sigma=\int_0^\infty e^{-x^2}\sqrt{\cosh^2(x)-1}\,dx=\int_0^\infty e^{-x^2}\sinh(x)\,dx$$

$$=\int_0^\infty \frac{e^{-x^2}}{2}(e^x-e^{-x})dx=\frac{1}{2}(\int_0^\infty e^{-x^2+x}\,dx-$$

$$-\int_0^\infty e^{-x^2-x}dx)=\frac{1}{2}(\int_0^\infty e^{-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx-\int_0^\infty e^{-\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}dx)=$$

$$=\frac{1}{2}e^{\frac{1}{4}}\left(\int_0^\infty e^{-\left(x-\frac{1}{2}\right)^2}dx-\int_0^\infty e^{-\left(x+\frac{1}{2}\right)^2}dx\right)=\frac{e^{\frac{1}{4}}}{2}(\sigma_1-\sigma_2)$$

$$\sigma_1=\int_0^\infty e^{-\left(x-\frac{1}{2}\right)^2}dx=\frac{\sqrt{\pi}}{2}[\,\text{erf}\left(x-\frac{1}{2}\right)]_0^\infty=\frac{\sqrt{\pi}}{2}\left[\text{erf}(\infty)-\text{erf}\left(\frac{1}{2}\right)\right]=\frac{\sqrt{\pi}}{2}\left(1+\text{erf}\left(\frac{1}{2}\right)\right)$$

$$\sigma_2=\int_0^\infty e^{-\left(x+\frac{1}{2}\right)^2}dx=\frac{\sqrt{\pi}}{2}[\text{erf}\left(x+\frac{1}{2}\right)]_0^\infty=\frac{\sqrt{\pi}}{2}\left(\text{erf}\,(\infty)-\text{erf}\,\left(\frac{1}{2}\right)\right)=\frac{\sqrt{\pi}}{2}\left(1-\text{erf}\,\left(\frac{1}{2}\right)\right)$$

$$\sigma=\int_0^\infty e^{-x^2}\sqrt{\cosh^2(x)-1}dx=\frac{e^{\frac{1}{4}}}{2}(\sigma_1-\sigma_2)=\frac{e^{\frac{1}{4}}\sqrt{\pi}}{2}\text{erf}\left(\frac{1}{2}\right)$$

Notes:

Error function: $\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-z^2}dz=\text{erf}\,[z];\ \text{erf}(\pm\infty)=\pm 1;\ \text{erf}\,(\pm a)=\pm\,\text{erf}\,(a)$