ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^\infty\int_0^\infty\int_0^\infty \frac{\log_e\left(\frac{1}{(x+y+z)^{x+y+z}}\right)}{1+e^{x+y+z}}dxdydz$$

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$$*\,\Omega=-\int_0^\infty\int_0^\infty\int_0^\infty \frac{(x+y+z)\ln(x+y+z)}{1+e^{x+y+z}}dxdydz$$

$$Let: x\to e^{-x}, y\to e^{-y}, z\to e^{-z}\to\Omega$$

$$=-\int_0^1\int_0^1\int_0^1 \frac{\ln(xyz)\ln(-\ln(xyz))}{1+xyz}dxdydz$$

$$symmetry\to \frac{1}{2}\int_0^1\frac{\ln^3(x)\ln(-\ln(x))}{1+x}dx=-\frac{1}{2}\int_0^\infty\frac{x^3\ln(x)}{1+e^{-x}}e^{-x}dx$$

$$=-\frac{1}{2}\int_0^\infty x^3\ln(x)e^{-x}\left(\sum_{n=0}^\infty(-1)^n e^{-nx}\right)dx$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\int_0^\infty x^3\ln(x)e^{-x(n+1)}dx=$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\frac{d}{ds}|_{s=4}\int_0^\infty x^{s-1}e^{-(n+1)x}\,dx$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\frac{d}{ds}|_{s=4}M\{e^{-(n+1)x}\}(s)=$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\frac{d}{ds}|_{s=4}\frac{\Gamma(s)}{(n+1)^s}$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\left(\frac{\psi(s)\Gamma(s)}{(n+1)^s}-\frac{\ln(n+1)\,\Gamma(s)}{(n+1)^s}\right)|_{s=4}=$$

$$-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\left(\frac{-6\ln(n+1)-6\gamma+11}{(n+1)^4}\right)=$$

$$=-\frac{11}{2}\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^4}+3\gamma\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^4}+3\sum_{n=0}^\infty\frac{(-1)^n\ln(n+1)}{(n+1)^4}=$$

$$=-\frac{11}{2}\frac{7\pi^4}{720}+3\gamma\frac{7\pi^4}{720}+3\left(-\frac{7\zeta'(4)}{8}-\frac{\pi^4}{720}\ln(2)\right)=$$

$$=\frac{7\pi^4}{240}\gamma-\frac{77\pi^4}{1440}-\frac{21\zeta'(4)}{8}-\frac{\pi^4}{240}\ln(2)$$

notes:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^s}=\zeta(s)(1-2^{1-s})\to\frac{d}{ds}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^s}=\frac{d}{ds}\zeta(s)(1-2^{1-s})$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\ln(n+1)}{(n+1)^s}=-2^{-s}\big((2^s-2)\zeta'(s)+2\ln(2)\,\zeta(s)\big)$$