# ROMANIAN MATHEMATICAL MAGAZINE

**Prove that:**

$$\int_0^1\int_0^1 \frac{x^2y^2+4xy+1}{x^4y^4-x^4-y^4+8x^3y^3+16x^2y^2+8xy+1}dxdy=\frac{3G}{4}$$

*Proposed by Amin Hajiyev-Azerbaijan*

*Solution 1 by proposer*

$$I=\int_0^1\int_0^1 \frac{x^2y^2+4xy+1}{x^4y^4-x^4-y^4+8x^3y^3+16x^2y^2+8xy+1}dxdy=\int_0^1\int_0^1 f(x;y)dxdy$$

$$f(x;y)=\frac{x^2y^2+4xy+1}{x^4y^4-x^4-y^4+8x^3y^3+16x^2y^2+8xy+1}=$$

$$=\frac{x^2y^2+4xy+1}{(x^2y^2)^2+(4xy)^2+1+2(4xy)(x^2y^2)+2(4xy)+2x^2y^2-x^4-y^4-2x^2y^2}=$$

$$=\frac{x^2y^2+4xy+1}{(x^2y^2+4xy+1)^2-(x^2+y^2)^2}$$

$$=\frac{x^2y^2+4xy+1+x^2y^2+4xy+1+y^2+x^2-x^2-y^2}{2(x^2y^2+4xy+1+x^2+y^2)(x^2y^2+4xy+1-x^2-y^2)}=$$

$$=\frac{1}{2}\left(\frac{1}{x^2y^2+4xy+1+x^2+y^2}+\frac{1}{x^2y^2+4xy+1-x^2-y^2}\right)$$

$$I=\frac{1}{2}(I_1+I_2)\quad \left\{\begin{matrix} I_1=\int_0^1\int_0^1 \frac{1}{x^2y^2+4xy+1+x^2+y^2}dxdy \\ I_2=\int_0^1\int_0^1 \frac{1}{x^2y^2+4xy+1-x^2-y^2}dxdy \end{matrix}\right\}$$

$$I_1=\int_0^1\int_0^1 \frac{1}{x^2y^2+4xy+1+x^2+y^2}dxdy=\int_0^1\int_0^1 \frac{1}{(1+xy)^2+(x+y)^2}dxdy=$$

$$=\int_0^1\int_0^1 \frac{1}{(1+xy)^2\left(1+\left(\frac{x+y}{1+xy}\right)^2\right)}dxdy$$

$$let:\left\{\frac{x+y}{1+xy}=t, x=\frac{t-y}{1-yt}, \frac{dx}{dt}=\frac{1-y^2}{(1-yt)^2}, 1+xy=\frac{1-y^2}{1-yt};\ t[1;\ y]\right\}$$

$$I_1=\int_0^1\int_y^1 \frac{1-y^2}{\left(\frac{1-y^2}{1-yt}\right)^2(1+t^2)(1-yt)^2}dtdy=\int_0^1\int_y^1 \frac{1}{(1+t^2)(1-y^2)}dtdy$$

$$=\int_0^1 \frac{1}{1-y^2}[\tan^{-1}(t)]_y^1 dy=$$

$$=\frac{\pi}{4}\int_0^1 \frac{1}{1-y^2}dy-\int_0^1 \frac{\tan^{-1}(y)}{1-y^2}dy.\ \rightarrow\quad IBP\quad \left\{\begin{matrix} v=\int \frac{1}{1-y^2}dy=\ln\left(\frac{1+y}{1-y}\right). \\ u=\tan^{-1}(y), \frac{du}{dy}=\frac{1}{1+y^2} \end{matrix}\right\}$$

$$I_1=\frac{\pi}{4}\int_0^1\frac{1}{1-y^2}dy-\frac{\pi}{4}\int_0^1\frac{1}{1-y^2}dy+\frac{1}{2}\int_0^1\frac{\ln\left(\frac{1+y}{1-y}\right)}{1+y^2}dy,\left\{\frac{1-y}{1+y}=\theta;\quad \theta[0;1]\right\}$$

$$I_1=-\frac{1}{2}\int_0^1\frac{\ln(\theta)}{1+\theta^2}d\theta=-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1\theta^{2n}\ln(\theta)d\theta=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{G}{2}$$

$$I_2=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1-x^2-y^2}dxdy=\int_0^1\int_0^1\frac{1}{(1+xy)^2-(x-y)^2}dxdy$$

$$=\int_0^1\int_0^1\frac{1}{(1+xy)^2\left(1-\left(\frac{x-y}{1+xy}\right)^2\right)}dxdy$$

$$let:\left\{\frac{x-y}{1+xy}=m,x=\frac{m+y}{1-my},\frac{dx}{dm}=\frac{1+y^2}{(1-my)^2},1+xy=\frac{1+y^2}{1-my},m\left[\frac{1-y}{1+y};-y\right]\right\}$$

$$I_2=\int_0^1\int_{-y}^{\frac{1-y}{1+y}}\frac{1+y^2}{\left(\frac{1+y^2}{1-my}\right)^2(1-m^2)(1-my)^2}dmdy=\int_0^1\int_{-y}^{\frac{1-y}{1+y}}\frac{1}{(1+y^2)(1-m^2)}dmdy$$

$$=\int_0^1\frac{1}{1+y^2}[\operatorname{arctanh}(m)]_{-y}^{\frac{1-y}{1+y}}dy=$$

$$=\int_0^1\frac{\operatorname{arctanh}\left(\frac{1-y}{1+y}\right)}{1+y^2}dy+\int_0^1\frac{\operatorname{arctanh}(y)}{1+y^2}dy=2\int_0^1\frac{\operatorname{arctanh}(y)}{1+y^2}dy=-\int_0^1\frac{\ln\left(\frac{1-y}{1+y}\right)}{1+y^2}dy=$$

$$=-\int_0^1\frac{\ln(y)}{1+y^2}dy=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G$$

$$I=\frac{1}{2}(I_1+I_2)=\frac{1}{2}\left(G+\frac{G}{2}\right)=\frac{3G}{4}$$

*Solution 2 by Pham Duc Nam-Vietnam*

$$*\ x^4y^4+8x^3y^3+16x^2y^2+8xy-x^4-y^4=$$

$$=\left(x^2y^2\right)^2+16x^2y^2+1++2x^2y^2\bullet 4xy+2\bullet 4xy\bullet 1+2\bullet x^2y^2\bullet 1-\left(x^4+y^4+2x^2y^2\right)=$$

$$=\left(x^2y^2+4xy+1\right)^2-\left(x^2+y^2\right)^2$$

$$=\left(x^2y^2+4xy+1+\left(x^2+y^2\right)\right)\left(x^2y^2+4xy+1-\left(x^2+y^2\right)\right)$$

$$I=\frac{1}{2}\left(\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1+x^2+y^2}dxdy+\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1-x^2-y^2}dxdy\right)=\frac{1}{2}(J+K)$$

$$J=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1+(x^2+y^2)}dxdy=\int_0^1\left[\frac{\arctan\left(\frac{x+y}{1+xy}\right)}{1-y^2}\right]_0^1dy$$

$$=\int_0^1\frac{1}{1-y^2}\left(\frac{\pi}{4}-\arctan(y)\right)dy=$$

$$K=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1-(x^2+y^2)}dxdy=\frac{1}{2}\int_0^1\frac{\ln\left(\frac{xy+x-y+1}{xy-x+y+1}\right)}{1+y^2}\Big]_0^1 dy=$$

$$=\frac{1}{2}\int_0^1\frac{1}{1+y^2}\left(\ln\left(\frac{1+y}{1-y}\right)-\ln(y)\right)dy$$

$$J=\int_0^1\frac{1}{1-y^2}\left(\frac{\pi}{4}-\arctan(y)\right)dy,\ let:\left\{t=\frac{1-y}{1+y},y=\frac{1-t}{1+t},dy=-\frac{2}{(1+t)^2}dt\right\}$$

$$J=2\int_0^1\frac{1}{\frac{4t(1+t)^2}{(1+t)^2}}\left(\frac{\pi}{4}-arctan\left(\frac{1-t}{1+t}\right)\right)dt=\frac{1}{2}\int_0^1\frac{1}{t}\left(\frac{\pi}{4}-\frac{\pi}{4}+\arctan(t)\right)dt=$$

$$=\frac{1}{2}\int_0^1\frac{\arctan(t)}{t}dt=\frac{G}{2}$$

$$K=\frac{1}{2}\int_0^1\frac{1}{1+y^2}\left(\ln\left(\frac{1+y}{1-y}\right)-\ln(y)\right)dy=-\frac{1}{2}\int_0^1\frac{\ln(y)}{1+y^2}dy+\frac{1}{2}\int_0^1\frac{\ln\left(\frac{1+y}{1-y}\right)}{1+y^2}dy$$

$$=\frac{G}{2}-\frac{1}{2}\int_0^1\frac{\ln\left(\frac{1-y}{1+y}\right)}{1+y^2}dy,\qquad let:\left\{t=\frac{1-y}{1+y},y=\frac{1-t}{1+t},dy=-\frac{2}{(1+t)^2}dt\right\}$$

$$k=\frac{G}{2}-\frac{1}{2}\int_0^1\frac{\ln(t)}{1+t^2}dt=\frac{G}{2}+\frac{G}{2}=G$$

$$I=\frac{1}{2}(J+K)=\frac{1}{2}\left(G+\frac{G}{2}\right)=\frac{3G}{4}$$

***Notes***:

$$\int_0^1\frac{\arctan(t)}{t}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 t^{2n}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\beta(2)=G$$

$$\int_0^1\frac{\ln(t)}{1+t^2}dt=\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n}\ln(t)dt=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-G;\ arctan\left(\frac{1-t}{1+t}\right)$$

$$=\frac{\pi}{4}-arctan(t)$$