ROMANIAN MATHEMATICAL MAGAZINE

Prove that

$$I=\int_0^1\int_0^1\frac{y\log(1+xy^2)}{1-xy^2}dx\,dy=\int_0^1\int_0^1\frac{x\log(2-x^2)}{1-x^2y}dx\,dy=\frac{a}{b}\varsigma(b)\log(\sqrt[b]{b})-\frac{1}{b}\varsigma(a)$$

For: $a,b\in Z^+$ and $GCD(a,b)=1$. Find a,b.

Proposed by Bui Hong Suc-Vietnam

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\frac{y\log(1+xy^2)}{1-xy^2}dx\,dy\Bigg]_{xy^2=m}=\frac{1}{2}\int_0^1\frac{1}{x}\int_0^x\frac{\log(1+m)}{1-m}dm\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\frac{1}{2}\left[\log(x)\int_0^x\frac{\log(1+m)}{1-m}dm\Bigg]_{x=0}^{x=1}-\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx\right]=$$

$$=-\frac{1}{2}\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx=-\frac{1}{2}\left\{\varsigma(3)-\frac{\pi^2}{4}\log(2)\right\}=-\frac{1}{2}\varsigma(3)+\frac{\pi^2}{8}\log(2)$$

$$=-\frac{1}{2}\varsigma(3)+\frac{3}{4}\varsigma(2)\log(2)=-\frac{1}{2}\varsigma(3)+\frac{3}{2}\varsigma(2)\log(\sqrt{2})\Rightarrow a=3;b=2$$

$$\text{Note: }\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx=\varsigma(3)-\frac{\pi^2}{4}\log(2)$$

$$I=\int_0^1\int_0^1\frac{x\log(2-x^2)}{1-x^2y}dx\,dy=-\int_0^1\frac{\log(1-x^2)\log(2-x^2)}{x}dx\Bigg]_{x^2\to x}$$

$$=-\frac{1}{2}\int_0^1\frac{\log(1-x)\log(2-x)}{x}dx\Bigg]_{x\to 1-x}=-\frac{1}{2}\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx$$

$$=-\frac{1}{2}\left\{\varsigma(3)-\frac{\pi^2}{4}\log(2)\right\}=-\frac{1}{2}\varsigma(3)+\frac{\pi^2}{8}\log(2)$$

$$=-\frac{1}{2}\varsigma(3)+\frac{3}{4}\varsigma(2)\log(2)=-\frac{1}{2}\varsigma(3)+\frac{3}{2}\varsigma(2)\log(\sqrt{2})\Rightarrow a=3;b=2$$