ROMANIAN MATHEMATICAL MAGAZINE

**Find a closed form:**

$$\Omega=\int_0^1\int_0^1\frac{\ln(1-xy)\,Li_4(1-x)}{x(1-x)(1-xy)}dx\,dy$$

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*Solution by Togrul Ehmedov-Azerbaijan*

$$\Omega=\int_0^1\int_0^1\frac{\ln(1-xy)\,Li_4(1-x)}{x(1-x)(1-xy)}dx\,dy=-\frac{1}{2}\int_0^1\frac{\ln^2(1-x)\,Li_4(1-x)}{x^2(1-x)}dx$$

$$=-\frac{1}{2}\int_0^1\frac{\ln^2(x)\,Li_4(x)}{x(1-x)^2}dx$$

$$=-\frac{1}{2}\left\{\int_0^1\frac{\ln^2(x)\,Li_4(x)}{x}dx+\int_0^1\frac{\ln^2(x)\,Li_4(x)}{1-x}dx+\int_0^1\frac{\ln^2(x)\,Li_4(x)}{(1-x)^2}dx\right\}$$

$$\Omega_1=\int_0^1\frac{\ln^2(x)\,Li_4(x)}{x}dx=\sum_{k=1}^{\infty}\frac{1}{k^4}\int_0^1 x^{k-1}\ln^2(x)\,dx=2\sum_{k=1}^{\infty}\frac{1}{k^7}=2\varsigma(7)$$

$$\Omega_2=\int_0^1\frac{\ln^2(x)\,Li_4(x)}{1-x}dx=2\varsigma(3)\varsigma(4)+20\varsigma(2)\varsigma(5)-36\varsigma(7)$$

$$\Omega_3=\int_0^1\frac{\ln^2(x)\,Li_4(x)}{(1-x)^2}dx\overset{IBP}{=}-\int_0^1\frac{Li_3(x)\ln^2(x)}{x(1-x)}dx-2\int_0^1\frac{Li_4(x)\ln(x)}{x(1-x)}dx=$$

$$=-\int_0^1\frac{Li_3(x)\ln^2(x)}{x}dx-\int_0^1\frac{Li_3(x)\ln^2(x)}{1-x}dx-2\int_0^1\frac{Li_4(x)\ln(x)}{x}dx-2\int_0^1\frac{Li_4(x)\ln(x)}{1-x}dx$$

$$\Omega_{3a}=\int_0^1\frac{Li_3(x)\ln^2(x)}{x}dx=\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^1 x^{k-1}\ln^2(x)\,dx=2\sum_{k=1}^{\infty}\frac{1}{k^6}=2\varsigma(6)$$

$$\Omega_{3b}=\int_0^1\frac{Li_3(x)\ln^2(x)}{1-x}dx=\varsigma^2(3)-\varsigma(6)$$

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$$\Omega_{3c}=\int_0^1 \frac{Li_4(x)\ln(x)}{x}dx=\sum_{k=1}^{\infty}\frac{1}{k^4}\int_0^1 x^{k-1}\ln(x)\,dx=-\sum_{k=1}^{\infty}\frac{1}{k^6}=-\zeta(6)$$

$$\Omega_{3d}=\int_0^1 \frac{Li_4(x)\ln(x)}{1-x}dx=\zeta^2(3)-\frac{25}{12}\zeta(6)$$

$$\Omega_3=\Omega_{3a}-\Omega_{3b}-2\Omega_{3c}-2\Omega_{3d}=\frac{31}{6}\zeta(6)-3\zeta^2(3)$$

$$\Omega=-\frac{1}{2}\{\Omega_1+\Omega_2+\Omega_3\}=17\zeta(7)-\zeta(3)\zeta(4)-10\zeta(2)\zeta(5)-\frac{31}{12}\zeta(6)+\frac{3}{2}\zeta^2(3)$$