ROMANIAN MATHEMATICAL MAGAZINE

$$Let\ be\ f,g:[2;+\infty)\to R^+:f^3(x))=x+3f(x),f(2)=2$$

$$and\ \ g(x+1)+4x+3=g(x)+4x^3+6x^2,g(2)=2$$

$$Find:\ \Omega=\frac{\int_2^{2024}g[f(x)]}{\int_2^{2024}f[g(x)]}$$

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The solution of the functional equation:

$$g(x+1)+4x+3=g(x)+4x^3+6x^2,g(2)=2$$

is the polynomial

$$g(x)=ax^4+bx^3+cx^2+dx+e$$

Let's find the coefficients a,b,c,d and e from here.Because of this

$$g(x+1)-g(x)=a((x+1)^4-x^4)+b((x+1)^3-x^3)+c((x+1)^2-x^2)+$$
$$+d((x+1)-x)=4ax^3+(6a+3b)x^2+(4a+3b+2c)x+(a+b+c+d)=$$
$$=4x^3+6x^2-4x-3$$

From here

$$\begin{cases}4a=4\\6a+3b=6\\4a+3b+2c=-4\\a+b+c+d=-3\end{cases}\Rightarrow a=1;b=0;c=-4;d=0$$

We have

$$g(x)=x^4-4x^2+e\ ,g(2)=2\Rightarrow g(x)=x^4-4x^2+2=(x^2-2)^2-2$$
$$g(x)=(x^2-2)^2-2$$

The solution of the functional equation

$$f^3(x))=x+3f(x),f(2)=2$$

is

$$f(x)=2$$

So ,

$$g(x)=(x^2-2)^2-2\ \ ,f(x)=2$$

Therefore

$$g(f(x))=(f^2(x)-2)^2-2=(4-2)^2-2=2$$
$$f(g(x))=2$$

Then

$$\Omega=\frac{\int_2^{2024}g[f(x)]}{\int_2^{2024}f[g(x)]}=1$$