ROMANIAN MATHEMATICAL MAGAZINE

**Find a closed form:**

$$\int_{-\infty}^{\infty}\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)(1+y+y^2)}dxdy$$

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*Solution by Shirvan Tahirov-Azerbaijan*

$$\int_{-\infty}^{\infty}\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)(1+y+y^2)}dxdy=\int_{-\infty}^{\infty}\frac{dy}{(1+y+y^2)^2}\cdot\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)}dx$$
$$=Y\,.\;X$$

$$Y=\int_{-\infty}^{\infty}\frac{dy}{(1+y+y^2)^2}=\int_{-\infty}^{\infty}\frac{dy}{(\frac{1}{4}+y+y^2+\frac{3}{4})^2}=\int_{-\infty}^{\infty}\frac{dy}{(\left(\frac{1}{2}+y\right)^2+\frac{3}{4})^2}$$

$$\underbrace{\overbrace{=}^{y+1=\frac{\sqrt{3}}{2}\tan(z)}}_{dy=\frac{\sqrt{3}}{2}sec^2(z)}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\frac{\sqrt{3}}{2}sec^2(z)}{((\frac{\sqrt{3}}{2}\sec(z))^2)^2}dz=\frac{\sqrt{3}}{2}.\frac{16}{9}.2\int_{0}^{\frac{\pi}{2}}cos^2(z)dz=\frac{4\pi}{3\sqrt{3}}$$

$$X=\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)}dx\overset{x\to\frac{1}{x}}{\cong}-\int_{0}^{1}\frac{xln^3(x)}{(1-x)^2}dx=-\sum_{n=0}^{\infty}n\int_{0}^{1}x^n ln^3(x)dx$$

$$6\sum_{n=0}^{\infty}\frac{n}{(n+1)^4}=6\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)^3}-\frac{1}{(n+1)^4}\right)=6\sum_{n=0}^{\infty}\frac{1}{(n+1)^3}-6\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}$$

$$=6\zeta(3)-6\zeta(4)=6(\zeta(3)-\frac{\pi^4}{90})$$

$$\int_{-\infty}^{\infty}\frac{dy}{(1+y+y^2)^2}\cdot\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)}dx\;=Y.X=\frac{8\pi}{\sqrt{3}}(\zeta(3)-\frac{\pi^4}{90})$$