ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\ln(xyz)}{1-(xyz)^2}dxdydz$$

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$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\ln(xyz)}{1-x^2y^2z^2}dxdydz,\left\{\int_0^1\int_0^1 f(xy)dxdy=-\int_0^1\ln(x)f(x)dx\right\}$$

$$\Omega=-\int_0^1\int_0^1\frac{x^2z\ln(xz)ln(x)}{1-x^2z^2}dxdz$$

$$=-\sum_{n=0}^{\infty}\int_0^1\int_0^1 x^{2n+2}z^{2n+1}\left(\ln^2(x)+\ln(x)\ln(z)\right)dxdz=$$

$$=-\sum_{n=0}^{\infty}\int_0^1 z^{2n+1}dz\int_0^1 x^{2n+2}\ln^2(x)\,dx-\sum_{n=0}^{\infty}\int_0^1 z^{2n+1}\ln(z)dz\int_0^1 x^{2n+2}\ln(x)\,dx=$$

$$=-2\sum_{n=0}^{\infty}\frac{1}{(2n+2)(2n+3)^3}-\sum_{n=0}^{\infty}\frac{1}{(2n+2)^2(2n+3)^2}=$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2n+3)^2}+2\sum_{n=0}^{\infty}\frac{1}{(2n+3)^3}-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$$

$$\text{we have.}\left\{\sum_{n=1}^{\infty}a_n=\sum_{n=0}^{\infty}a_{2n+1}+\sum_{n=1}^{\infty}a_{2n}\right\}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{(n+2)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n+2)^2}+2\sum_{n=1}^{\infty}\frac{1}{(n+2)^3}-2\sum_{n=1}^{\infty}\frac{1}{(2n+2)^3}-\frac{\zeta(2)}{4}=$$

$$=\frac{\pi^2}{6}-\frac{5}{4}-\frac{\pi^2}{24}+\frac{1}{4}+2\zeta(3)-\frac{9}{4}-\frac{1}{4}\zeta(3)+\frac{1}{4}-\frac{\pi^2}{24}=$$

$$=\frac{\pi^2}{12}+\frac{7\zeta(3)}{4}-3$$

$$\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\ln(xyz)}{1-x^2y^2z^2}dxdydz=\frac{\pi^2}{12}+\frac{7\zeta(3)}{4}-3$$