ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\log(xyz)}{1-x^2y^2z^2}\,dx\,dy\,dz$$

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$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\log(xyz)}{1-x^2y^2z^2}\,dx\,dy\,dz=\sum_{k=0}^{\infty}\int_0^1\int_0^1\int_0^1 xy(xyz)^{2k+1}\log(xyz)\,dx\,dy\,dz$$

Let $xyz=m$

$$\Omega=\sum_{k=0}^{\infty}\int_0^1\int_0^1\int_0^{xy} m^{2k+1}\log(m)\,dm\,dy\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\sum_{k=0}^{\infty}\int_0^1\left\{\left[y\int_0^{xy} m^{2k+1}\log(m)\,dm\right]_{y=0}^{y=1}-\int_0^1(xy)^{2k+2}\log(xy)\,dy\right\}dx=$$

$$=\sum_{k=0}^{\infty}\int_0^1\int_0^x m^{2k+1}\log(m)\,dm\,dx-\sum_{k=0}^{\infty}\int_0^1\int_0^1(xy)^{2k+2}\log(xy)\,dy\,dx=$$

$$=\sum_{k=0}^{\infty}\left\{\left[x\int_0^x m^{2k+1}\log(m)\,dm\right]_{x=0}^{x=1}-\int_0^1 x^{2k+2}\log(x)\,dx\right\}$$

$$-\sum_{k=0}^{\infty}\int_0^1\frac{1}{x}\int_0^x p^{2k+2}\log(p)\,dp\,dx=$$

$$=\sum_{k=0}^{\infty}\int_0^1 m^{2k+1}\log(m)\,dm-\sum_{k=0}^{\infty}\int_0^1 x^{2k+2}\log(x)\,dx+\sum_{k=0}^{\infty}\int_0^1 x^{2k+2}\log^2(x)\,dx=$$

$$=-\sum_{k=0}^{\infty}\frac{1}{(2k+2)^2}+\sum_{k=0}^{\infty}\frac{1}{(2k+3)^2}+2\sum_{k=0}^{\infty}\frac{1}{(2k+3)^3}=$$

$$=-\frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}+\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}-1\right)+2\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}-1\right)=$$

$$=-\frac{\pi^2}{24}+\left(\frac{\pi^2}{8}-1\right)+2\left(\frac{7}{8}\varsigma(3)-1\right)=\frac{\pi^2}{12}+\frac{7}{4}\varsigma(3)-3=\frac{1}{2}\varsigma(2)+\frac{7}{4}\varsigma(3)-3$$