ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\xi=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\frac{\sqrt{xy}}{\sqrt{yz}+\sqrt{x}}dxdydz$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Abbaszade Yusif-Azerbaijan

$$\xi=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\frac{\sqrt{xy}}{\sqrt{yz}+\sqrt{x}}dxdydz\underset{symmetry}{\overset{by}{=}}3\int_0^1\int_0^1\int_0^1\frac{\sqrt{xy}}{\sqrt{yz}+\sqrt{x}}dxdydz$$

$$a^2=x,b^2=y,c^2=z\qquad dx=2ada,dy=2bdb,dz=2cdc\qquad a,b,c\in[0,1]$$

$$\xi=3\int_0^1\int_0^1\int_0^1\frac{a^2b^2c}{a+bc}\times 8dadbdc$$

$$=24\int_0^1\int_0^1\int_0^1\frac{a^2b^2c}{a+bc}dadbdc\qquad\{a+bc=t,bdc=dt\}$$

$$\xi=24\int_0^1\int_0^1\int_a^{a+b}\frac{a^2b^2}{t}\times\frac{t-a}{b}\times\frac{dt}{b}dadb=24\int_0^1\int_0^1\int_a^{a+b}\frac{a^2(t-a)}{t}dtdadb$$

$$\xi=24\int_0^1\int_0^1\left(a^2(a+b-a)-a^3\ln\left(1+\frac{b}{a}\right)\right)dadb=$$

$$=24\left(\int_0^1\int_0^1a^2bdadb-\int_0^1a^3\ln\left(1+\frac{1}{a}\right)da+\int_0^1\int_0^1\frac{a^3b}{a+b}dadb\right)\ \{a+b=t\}$$

$$\xi=24\left(\frac{1}{6}-\frac{a^4}{4}\ln\left(1+\frac{1}{a}\right)_0^1+\int_0^1\frac{a^4}{4}\left(\frac{1}{1+a}-\frac{1}{a}\right)da+\int_0^1\int_a^{a+1}\frac{a^3}{t}\times(t-a)dtda\right)$$

$$\xi=24\left(\frac{1}{6}-\frac{1}{4}\ln(2)+\frac{1}{4}\ln(2)-\frac{1}{4}\times\frac{7}{12}-\frac{1}{16}+\int_0^1\left(a^3(a+1-a)-a^4\ln\left(1+\frac{1}{a}\right)\right)da\right)$$

$$\xi=24\left(\frac{1}{6}-\frac{7}{48}-\frac{1}{16}+\int_0^1a^3da-\frac{a^5}{5}\ln\left(1+\frac{1}{a}\right)_0^1+\int_0^1\frac{a^5}{5}\left(\frac{1}{1+a}-\frac{1}{a}\right)da\right)$$

$$\xi=24\left(-\frac{1}{24}+\frac{1}{4}-\frac{1}{5}\ln(2)+\frac{1}{5}\times\frac{47}{60}-\frac{1}{5}\ln(2)-\frac{1}{25}\right)$$

$$\xi=24\left(\frac{39}{120}-\frac{2}{5}\ln(2)\right)$$

$$\xi=\frac{39}{5}-\frac{48}{5}\ln(2)$$