ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^\infty \frac{e^{-x}(1-\cos(3x))}{x^2}dx$$

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$$Let\ \ f(t)=\int_0^\infty \frac{e^{-x}(1-\cos(tx))}{x^2}dx \to \frac{d}{dt}f(t)=\frac{d}{dt}\left(\int_0^\infty \frac{e^{-x}(1-\cos(tx))}{x^2}dx\right)$$

$$\frac{d}{dt}f(t)=\int_0^\infty \frac{\partial}{\partial t}\frac{e^{-x}(1-\cos(tx))}{x^2}dx=\int_0^\infty \frac{e^{-x}}{x^2}(x\sin(tx))\,dx=\int_0^\infty \frac{e^{-x}}{x}\sin(tx)\,dx$$

$$\frac{d^2}{dt^2}f(t)=\int_0^\infty e^{-x}\cos(tx)dx=\mathcal{R}e\left\{\int_0^\infty e^{-x+itx}\,dx\right\}=\mathcal{R}e\left\{\frac{e^{x(-1+it)}}{1-it}\Big]_0^\infty\right\}=\mathcal{R}e\left\{\frac{1}{1-it}\right\}$$

$$\frac{d^2}{dt^2}f(t)=\mathcal{R}e\left\{\frac{1+it}{1+t^2}\right\}=\frac{1}{1+t^2}\to\frac{d}{dt}f(t)=\tan^{-1}(t)+C\underset{t=0}{\longrightarrow}\frac{d}{dt}f(0)$$
$$=\tan^{-1}(0)+C_1$$

$$\int_0^\infty \frac{e^{-x}}{x}\sin(0\times x)\,dx=0=0+C_1\to C_1=0\to\frac{d}{dt}f(t)=\tan^{-1}(t)$$

$$f(t)=\int\tan^{-1}(t)dt\underbrace{=}_{IBP}t\tan^{-1}(t)-\frac{1}{2}\ln(t^2+1)+C_2\underset{t=0}{\longrightarrow}\ C_2=0$$

$$\int_0^\infty \frac{e^{-x}(1-\cos(tx))}{x^2}dx=t\tan^{-1}(t)-\frac{1}{2}\ln(t^2+1)\underset{t=3}{\longrightarrow}$$

$$\text{ANSWER}=3\tan^{-1}(3)-\frac{1}{2}\ln(10)$$