ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_1^{\sqrt{3}} \left(\frac{\tan^{-1} x}{x - \tan^{-1} x}\right)^2 dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Yen Tung Chung-Taiwan

$$\underbrace{\int_1^{\sqrt{3}} \left(\frac{\tan^{-1} x}{x - \tan^{-1} x}\right)^2 dx}_{let\ y=\tan^{-1}x \Rightarrow x=\tan y, dx=\sec^2 y dy} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{y}{\tan y - y}\right)^2 \sec^2 y\, dy =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{y^2}{(\sin y - y\cos y)^2} dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{y^2}{(1+y^2)\left(\frac{1}{\sqrt{1+y^2}}\sin y - \frac{y}{\sqrt{1+y^2}}\cos y\right)} dy =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1 - \frac{1}{1+y^2}}{(\cos(\tan^{-1} y)\sin y - \sin(\tan^{-1} y)\cos y)^2} dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin^2(y - \tan^{-1} y)} d(y - \tan^{-1} y) =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2(y - \tan^{-1} y)\, d(y - \tan^{-1} y) = -\cot(y - \tan^{-1} y)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1 + \tan y \tan(\tan^{-1} y)}{\tan(\tan^{-1} y) - \tan y}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1 + y\tan y}{1 - \tan y}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1 + \frac{\pi}{\sqrt{3}}}{\frac{\pi}{3} - \sqrt{3}} - \frac{1 + \frac{\pi}{4}}{\frac{\pi}{4} - 1} =$$

$$= \frac{\pi\sqrt{3} + 3}{\pi - 3\sqrt{3}} + \frac{4 + \pi}{4 - \pi}$$

Solution 2 by Pham Duc Nam-Vietnam

$$\int \frac{\arctan^2(x)}{(x - \arctan^2(x))^2} dx = \int \frac{(\arctan(x) - x)^2 - (x^2 - 2x\arctan(x))}{(\arctan(x) - x)^2} dx$$

$$= x - \int \frac{x^2 - 2x\arctan(x)}{(\arctan(x) - x)^2} dx = x - \int \frac{x^2\left(1 - \frac{2\arctan(x)}{x}\right)(1 + x^2)}{(1 + x^2)(\arctan(x) - x)^2} dx$$

$$\begin{cases} u = \left(1 - \dfrac{2\arctan(x)}{x}\right)(1+x^2) \\ dv = \dfrac{x^2}{1+x^2}\dfrac{1}{(\arctan(x)-x)^2}dx \end{cases} \Rightarrow \begin{cases} du = \dfrac{2(1-x^2)(\arctan(x)-x)}{x^2} \\ v = \dfrac{1}{\arctan(x)-x} \end{cases}$$

$$\Rightarrow \int \frac{\arctan^2(x)}{(x-\arctan^2(x))^2}dx =$$

$$= x - \frac{1}{\arctan(x)-x}\left(1 - \frac{2\arctan(x)}{x}\right)(1+x^2) + 2\int \frac{1-x^2}{x^2}dx$$

$$= \frac{x\arctan(x)+1}{\arctan(x)-x} + C \Rightarrow \int_1^{\sqrt{3}} \frac{\arctan^2(x)}{(x-\arctan^2(x))^2}dx =$$

$$= \frac{x\arctan(x)+1}{\arctan(x)-x}\Bigg|_1^{\sqrt{3}} = \frac{4+\pi}{4-\pi} - \frac{\pi\sqrt{3}+3}{3\sqrt{3}-\pi}$$

Solution 3 by Ravi Prakash-India

Put $\tan^{-1}x = \theta, x = \tan\theta$

$$\Omega = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\theta}{\tan\theta - \theta}\right)^2 \sec^2\theta\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x^2}{(\sin x - x\cos x)^2}dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x\sin x}{(\sin x - x\cos x)^2}\cdot\frac{x}{\sin x}dx$$

$$= \frac{-1}{\sin x - x\cos x}\cdot\frac{x}{\sin x}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin x - x\cos x}\frac{(\sin x - x\cos x)dx}{\sin^2 x}$$

$$= \frac{-1}{\sin x - x\cos x}\cdot\frac{x}{\sin x}\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{3}} - \cot x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{-1}{\frac{\sqrt{3}}{2} - \frac{\pi}{3}\cdot\frac{1}{2}}\cdot\frac{\frac{\pi}{3}}{\frac{\sqrt{3}}{2}} + \frac{1}{\frac{1}{\sqrt{2}} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}}\cdot\frac{\frac{\pi}{4}}{\frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{3}} + 1$$

$$= \frac{2\pi}{4-\pi} - \frac{4\pi}{9-\sqrt{3}\pi} + 1 - \frac{1}{\sqrt{3}} = \frac{\pi+4}{4-\pi} - \left[\frac{4\pi\sqrt{3}+9-\sqrt{3}\pi}{\sqrt{3}(9-\sqrt{3}\pi)}\right]$$

$$= \frac{\pi+4}{4-\pi} - \frac{9+3\sqrt{3}\pi}{3(3\sqrt{3}-\pi)} = \frac{\pi+4}{4-\pi} - \frac{3+\sqrt{3}\pi}{3\sqrt{3}-\pi}$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega = \int_1^{\sqrt{3}} \left(\frac{\tan^{-1} x}{x - \tan^{-1} x}\right)^2 dx$$

$$= \int_1^{\sqrt{3}} \left(1 + \frac{x^2}{(x - \tan^{-1}(x))^2} - \frac{2x}{(x - \tan^{-1}(x))}\right) dx$$

$$= \left(\sqrt{3} - 1\right) + \int_1^{\sqrt{3}} \frac{-x^2 + 2x \cdot \tan^{-1}(x)}{(x - \tan^{-1}(x))^2} dx \overset{say}{\longrightarrow} S$$

Note: $\frac{1}{(x - \tan^{-1}(x))} = u \to u' = \frac{-\frac{x^2}{1+x^2}}{(x - \tan^{-1}(x))^2}$

$$S = \left(\sqrt{3} - 1\right) + \int_1^{\sqrt{3}} \frac{\frac{x^2}{(1+x^2)}(1+x^2) - (1+x^2) \cdot (-\tan^{-1}(x) + x)}{(x - \tan^{-1}(x))^2} dx$$

$$= \left(\sqrt{3} - 1\right) + \int_1^{\sqrt{3}} \frac{-(\tan^{-1}(x) - x)' \cdot (1+x^2) + (1+x^2) \cdot (\tan^{-1}(x) - x)}{(x - \tan^{-1}(x))} dx$$

$$= \sqrt{3} - 1 + \left[\frac{x^2 + 1}{\tan^{-1}(x) - x}\right]_1^{\sqrt{3}} = \sqrt{3} - 1 + \frac{12}{\pi - 3\sqrt{3}} - \frac{8}{\pi - 4}$$

$$\Rightarrow \Omega = \sqrt{3} - 1 + \frac{12}{\pi - 3\sqrt{3}} - \frac{8}{\pi - 4}$$