ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{\log(x^2)\log(y^2)\log(z^2)\log(xyz)}{1+xyz}dxdydz$$

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$$\Omega=\int_0^1\int_0^1\int_0^1\frac{\ln(x^2)\ln(y^2)\ln(z^2)\,xyz}{1+xyz}dxdydz=$$

$$\lim_{a\to0}\frac{\partial}{\partial a}\int_0^1\int_0^1\int_0^1\frac{\ln(x^2)\ln(y^2)\ln(z^2)\,(xyz)^a}{1+xyz}dxdydz=$$

$$8\lim_{a\to0}\frac{\partial}{\partial a}\sum_{n\geq0}(-1)^n\int_0^1\int_0^1\int_0^1 x^a\ln(x)\,y^a\ln(y)\,z^a\ln(z)\,(xyz)^n dxdydz=$$

$$8\lim_{a\to0}\frac{\partial}{\partial a}\sum_{n\geq0}(-1)^n\int_0^1\int_0^1\int_0^1\ln(x)\ln(y)\ln(z)\,x^{a+n}y^{a+n}z^{a+n}dxdydz=$$

$$8\lim_{a\to0}\frac{\partial}{\partial a}\sum_{n\geq0}(-1)^n(\int_0^1 x^{a+n}\ln(x)\,dx+\int_0^1 y^{a+n}\ln(y)\,dx+\int_0^1 z^{a+n}\ln(z)\,dz)=$$

$$\Delta:\int_0^1 x^{a+n}\ln(x)\,dx\overset{I.B.P}{=}\left(\frac{\ln(x).x^{n+a+1}}{n+a+1}\right)_0^1$$

$$-\int_0^1\frac{x^{n+a}}{n+a+1}dx=0-\frac{1}{(n+a+1)}\int_0^1 x^{n+a}dx=-\frac{1}{(n+a+1)^2}$$

The other two integral's are solved in the same form ...

$$8\lim_{a\to0}\frac{\partial}{\partial a}\sum_{n\geq0}(-1)^n\left(\left(-\frac{1}{(n+a+1)^2}\right).\left(-\frac{1}{(n+a+1)^2}\right).\left(-\frac{1}{(n+a+1)^2}\right)\right)=$$

$$-8\lim_{a\to0}\frac{\partial}{\partial a}\sum_{n\geq0}\frac{(-1)^n}{(n+a+1)^6}=48\lim_{a\to0}\sum_{n\geq0}\frac{(-1)^n}{(n+a+1)^7}=48\sum_{n\geq0}\frac{(-1)^n}{(n+1)^7}=$$

$$=48\sum_{n\geq1}\frac{(-1)^{n+1}}{(n)^7}=48\eta(7)$$

Where $\eta(v)$ is the Dirichlet eta function.