ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$I=\int_e^{e^2}\int_e^{e^2}\frac{log_x(y^x)+log_y(x^y)}{\sqrt{xy}}dxdy$$

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$$I=\int_e^{e^2}\int_e^{e^2}\frac{\log_x(y^x)+\log_y(x^y)}{\sqrt{xy}}dxdy=\int_e^{e^2}\int_e^{e^2}\frac{\frac{x\ln(y)}{\ln(x)}+\frac{y\ln(x)}{\ln(y)}}{\sqrt{xy}}dxdy$$

$$I=\int_e^{e^2}\int_e^{e^2}\left(\frac{\sqrt{x}\ln(y)}{\sqrt{y}\ln(x)}+\frac{\sqrt{y}\ln(x)}{\sqrt{x}\ln(y)}\right)dxdy=_{symmetry}^{by}2\int_e^{e^2}\int_e^{e^2}\frac{\sqrt{x}\ln(y)}{\sqrt{y}\ln(x)}dxdy$$

$$=2\int_e^{e^2}\frac{\sqrt{x}}{\ln(x)}\int_e^{e^2}\frac{\ln(y)}{\sqrt{y}}dxdy=2\,I_1\times I_2$$

$$I_1=\int_e^{e^2}\frac{\sqrt{x}}{\ln(x)}dx\underset{\ln(x)=t}{\overset{t[1;2]}{=}}\int_1^2\frac{e^{\frac{3t}{2}}}{t}dt=Ei\left[\frac{3x}{2}\right]_1^2=Ei(3)-Ei\left(\frac{3}{2}\right)$$

$$I_2=\int_e^{e^2}\frac{\ln(y)}{\sqrt{y}}dy=^{IBP}\left[2\sqrt{y}\ln(y)\right]_e^{e^2}-\int_e^{e^2}\frac{2}{\sqrt{y}}dy=\left[2\sqrt{y}\ln(y)-4\sqrt{y}\right]_e^{e^2}=2\sqrt{e}$$

$$I=2I_1\times I_2=4\sqrt{e}\left(Ei(3)-Ei\left(\frac{3}{2}\right)\right)$$

Note section :

$$\int\left(\frac{e^{ax}}{x}\right)dx=Ei(ax)+c$$