ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_0^{\infty} \frac{dx}{x^4+x^2+1}$$

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$$\frac{dx}{x^4+x^2+1} = \frac{1}{(x^2+1)^2-x^2} = \frac{1}{(x^2-x+1)(x^2+x+1)} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+x+1} =$$

$$\frac{(A+C)x^3+(A+B-C+D)x^2+(A+B+C-D)x+(B+D)}{(x^2-x+1)(x^2+x+1)}$$

$$\begin{cases} A+C=0 \\ A+B-C+D=0 \\ A+B+C-D=0 \\ B+D=0 \end{cases} \to \begin{cases} A+C=0 \\ A+B=0 \\ B+D=0 \\ C-D=0 \end{cases} \to \begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \\ C=\frac{1}{2} \\ D=\frac{1}{2} \end{cases}$$

$$\frac{1}{x^4+x^2+1} = \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2-x+1} + \frac{\frac{1}{2}x+\frac{1}{2}}{x^2+x+1}$$

$$\int_0^{\infty} \frac{dx}{x^4+x^2+1} = -\frac{1}{2}\int_0^{\infty} \frac{x-1}{x^2-x+1}dx + \frac{1}{2}\int_0^{\infty} \frac{x+1}{x^2+x+1}dx = -\frac{1}{4}\int_0^{\infty} \frac{2x-1}{x^2-x+1}dx +$$

$$\frac{1}{4}\int_0^{\infty} \frac{1}{x^2-x+1}dx + \int_0^{\infty} \frac{2x+1}{x^2+x+1}dx + \frac{1}{4}\int_0^{\infty} \frac{1}{x^2+x+1}dx =$$

$$-\frac{1}{4}\int_0^{\infty} \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{4}\int_0^{\infty} \frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{4}\int_0^{\infty} \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{4}\int_0^{\infty} \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} =$$

$$\frac{1}{4}(-\ln(x^2-x+1)+\ln(x^2+x+1)|_0^{\infty} + \frac{1}{4}(\int_0^{\infty} \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} + \int_0^{\infty} \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}) =$$

$$\frac{1}{4}\left(ln\frac{x^2+x+1}{x^2-x+1}\right)\Big|_0^{\infty} + \frac{1}{4}\cdot\frac{2}{\sqrt{3}}\left(arctan\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg|_0^{\infty} =$$

$$=\frac{1}{4}\lim_{x\to\infty}\left(\frac{x^2+x+1}{x^2-x+1}-ln1\right)+\frac{\sqrt{3}}{6}\left(\frac{\pi}{2}-\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{6}\right)=0+\frac{\sqrt{3}}{6}.\pi=\pi\frac{\sqrt{3}}{6}$$

$$\int_0^\infty \frac{dx}{x^4+x^2+1}=\pi\frac{\sqrt{3}}{6}\ (proved)$$