ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_{-1}^{1}\frac{dx}{x^2+x+1+\sqrt{x^4+3x^2+1}}$$

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$$\frac{1}{x^2+x+1+\sqrt{x^4+3x^2+1}}=\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{\left(x^2+x+1+\sqrt{x^4+3x^2+1}\right)\left(x^2+x+1-\sqrt{x^4+3x^2+1}\right)}=$$

$$\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{(x^2+x+1)^2-(x^4+3x^2+1)}=\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{x^4+x^2+1+2x^3+2x^2+2x-x^4-3x^2-1}=$$

$$\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{2x(x^2+1)}=\frac{1}{2x}+\frac{1}{2(x^2+1)}-\frac{\sqrt{x^4+3x^2+1}}{2x(x^2+1)}$$

$$\int_{-1}^{1}\left(\frac{1}{2x}+\frac{1}{2(x^2+1)}-\frac{\sqrt{x^4+3x^2+1}}{2x(x^2+1)}\right)dx=\frac{1}{2}\int_{-1}^{1}\frac{1}{x}dx+\frac{1}{2}\int_{-1}^{1}\frac{dx}{x^2+1}-\frac{1}{2}\int_{-1}^{1}\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}dx$$

$$f(x)=\frac{1}{x}\ ,\qquad g(x)=\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}\to f(x)\ and\ g(x)\ odd\ function$$

$$\int_{-1}^{1}\frac{1}{x}dx=0\ ,\qquad \int_{-1}^{1}\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}dx=0$$

$$\frac{1}{2}\int_{-1}^{1}\frac{1}{x}dx+\frac{1}{2}\int_{-1}^{1}\frac{dx}{x^2+1}-\frac{1}{2}\int_{-1}^{1}\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}dx=0+\frac{1}{2}\int_{-1}^{1}\frac{dx}{x^2+1}-0=\frac{\arctan(x)}{2}\bigg|_{-1}^{1}=$$

$$=\frac{1}{2}(\arctan(1)-\arctan(-1))=\frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\frac{\pi}{4}$$

$$\int_{-1}^{1}\frac{dx}{x^2+x+1+\sqrt{x^4+3x^2+1}}=\frac{\pi}{4}$$