ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\int_1^4 \frac{dx}{x^2+x\sqrt{x}}$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$Let\ \sqrt{x}=t\to dt=\frac{dx}{2\sqrt{x}},\qquad \int_1^2\frac{2t}{t^4+t^3}dt=2\int_1^2\frac{dt}{t^3+t^2}$$

$$\frac{1}{t^3+t^2}=\frac{1}{t^2(t+1)}=\frac{At+B}{t^2}+\frac{C}{t+1}$$

$$(At+B)(t+1)+Ct^2=1,\ \ At^2+At+Bt+B+Ct^2=1$$

$$(A+B)t^2+(A+B)t+B=1,\begin{cases}A+C=1\\A+B=0\\B=1\end{cases}\to\begin{cases}A=-1\\B=1\\C=1\end{cases}$$

$$Then:\ \frac{1}{t^3+t^2}=\frac{-t+1}{t^2}+\frac{1}{t+1}=-\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t+1}$$

$$2\int_1^2\frac{dt}{t^3+t^2}=2\int_1^2(-\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t+1})dt=2\left(\ln\left(\frac{t+1}{t}-\frac{1}{t}\right)\right)\Big|_2^1=$$

$$=2(\left(\ln\left(\frac{3}{2}\right)-\frac{1}{2}\right)-(\ln(2)-1))=2\ln(\left(\frac{3}{4}\right)+\frac{1}{2})=2\ln(\frac{3}{4})+1$$

$$\int_1^4\frac{dx}{x^2+x\sqrt{x}}=2\ln(\frac{3}{4})+1$$