ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_{-\pi}^{\pi} ln^2\left(\cos\left(\frac{x}{2}\right)\right)dx = 2\pi ln^2(2) + \frac{\pi^3}{6}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Shirvan Tahirov-Azerbaiajn

$$\int_{-\pi}^{\pi} ln^2\left(\cos\left(\frac{x}{2}\right)\right)dx = 4\int_0^1 \frac{ln^2(2)}{\sqrt{1-x^2}}dx = \int_0^1 \lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\frac{x^a}{\sqrt{1-x^2}}dx = \lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 \frac{x^a}{\sqrt{1-x^2}}dx =$$

$$\left\{x^2 = t, \quad \frac{dt}{dx} = 2x = 2\sqrt{t}\right\}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 \frac{t^{\frac{a}{2}}}{\sqrt{1-t}}\frac{dt}{2\sqrt{t}} = \frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 \frac{t^{\frac{a-1}{2}}}{\sqrt{1-t}}dt =$$

$$\frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 t^{\frac{a+1}{2}-1}(1-t)^{\frac{1}{2}-1}dt = \frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\beta\left(\frac{a+1}{2};\frac{1}{2}\right) = \frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\left(\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)}\right) =$$

$$\frac{\sqrt{\pi}}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\left(\frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)}\right) = \frac{\sqrt{\pi}}{2}\lim_{a\to 0}\frac{\Gamma\left(\frac{a+1}{2}\right)}{4\Gamma\left(\frac{a}{2}+1\right)}(\left(\psi^{(0)}\left(\frac{a}{2}+1\right) - \psi^{(0)}\left(\frac{a+1}{2}\right))\right)^2 - \psi^{(1)}(1) +$$

$$+\psi^{(1)}\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{8}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}(\left(\psi^{(0)}(1) - \psi^{(0)}\left(\frac{1}{2}\right))\right)^2 - \psi^{(1)}(1) + \psi^{(1)}\left(\frac{1}{2}\right) =$$

$$\frac{\sqrt{\pi}}{8}((\partial - (-\partial - 2\ln(2)))^2 - \frac{\pi^2}{6} + \frac{\pi^2}{2}) = 4.\frac{\pi}{8}(4ln^2(2) + \frac{\pi^2}{3}) = 2\pi ln^2(2) + \frac{\pi^3}{6}$$