ROMANIAN MATHEMATICAL MAGAZINE

Prove the integral relation:

$$\int_0^\infty e^{\frac{-\pi x}{3}}\sqrt[3]{\coth(3\pi x)+1}dx=\frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{13}{18}\right)}\int_0^\infty e^{\frac{-\pi 3}{3}}\sqrt[3]{\coth(3\pi x)-1}dx$$

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$$*I=\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)+1}dx=\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\frac{6^{6\pi x}+1}{e^{6\pi x}-1}+1}dx=$$

$$=\sqrt[3]{2}\int_0^\infty e^{\frac{5\pi x}{3}}\sqrt[3]{\frac{1}{e^{6\pi x}-1}}dx$$

$$t=\sqrt[3]{\frac{1}{e^{6\pi x}-1}}\Rightarrow x=\frac{1}{6\pi}\ln(1+t^3)\Rightarrow dx=-\frac{1}{2\pi}\frac{1}{t^4+t}dt$$

$$\Rightarrow I=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\left(\frac{1}{t^3}+1\right)^{\frac{5}{18}}\frac{1}{t^3+1}dt=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\frac{t^{-\frac{5}{6}}}{(t^3+1)^{\frac{13}{18}}}dt=$$

$$=\frac{\sqrt[3]{2}}{6\pi}\int_0^\infty\frac{(t^3)^{\frac{1}{18}-1}}{(t^3+1)^{\frac{1}{18}+\frac{12}{18}}}d(t^3)=\frac{\sqrt[3]{2}}{6\pi}B\left(\frac{1}{18},\frac{12}{18}\right)$$

$$*J=\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)-1}dx=\int_0^\infty 3^{-\frac{\pi x}{3}}\sqrt[3]{\frac{e^{6\pi x}+1}{e^{6\pi x-1}}-1}dx=$$

$$=\sqrt[3]{2}\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\frac{1}{e^{6\pi x}-1}}dx$$

$$t=\sqrt[3]{\frac{1}{e^{6\pi x}-1}}\Rightarrow x=\frac{1}{6\pi}\ln\left(1+\frac{1}{t^3}\right)\Rightarrow dx=-\frac{1}{2\pi}\frac{1}{t^4+t}dt$$

$$\Rightarrow I=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\left(\frac{1}{t^3}+1\right)^{-\frac{1}{18}}\frac{1}{t^3+1}dt=\frac{\sqrt[3]{2}}{2\pi}\int_0^\infty\frac{t^{\frac{1}{6}}}{(t^3+1)^{\frac{19}{18}}}dt=$$

$$=\frac{\sqrt[3]{2}}{6\pi}\int_0^{\infty}\frac{(t^3)^{\frac{7}{18}-1}}{(t^3+1)^{\frac{7}{18}+\frac{12}{18}}}d(t^3)=\frac{\sqrt[3]{2}}{6\pi}B\left(\frac{7}{18},\frac{12}{18}\right)$$

$$\Rightarrow\frac{I}{J}=\frac{\sqrt[3]{\frac{2}{6\pi}}B\left(\frac{1}{18},\frac{12}{18}\right)}{\frac{\sqrt[3]{2}}{6\pi}B\left(\frac{7}{18},\frac{12}{18}\right)}=\frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{12}{18}\right)}{\Gamma\left(\frac{13}{18}\right)}\frac{\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{12}{18}\right)}-\frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{13}{18}\right)}$$

$$\Rightarrow\int_0^{\infty}e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)+1}\,dx=\frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{13}{18}\right)}\int_0^{\infty}e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)-1}\,dx$$

Hence proved.