ROMANIAN MATHEMATICAL MAGAZINE

A Reciprocal integral relation: If we define the function $\psi(q)$

$$\psi(q)=\int_{-\infty}^{\infty}\frac{\tanh(4\pi x)-q}{\cosh^2(2\pi x)-q}dx$$

then prove the relation

$$\frac{\psi(q)}{q}=2q(q-1)\frac{\partial^2\psi(q)}{\partial q^2}+(2q+1)\frac{\partial\psi(q)}{\partial q}$$

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$$\psi(q)=\underbrace{\int_{-\infty}^{\infty}\frac{\tanh(4\pi x)}{\cosh^2(2\pi x)-q}}_{odd\ function}dx-q\int_{-\infty}^{\infty}\frac{dx}{\cosh^2(2\pi x)-q}$$

$$\psi(q)=2q\int_0^{\infty}\frac{dx}{\cosh^2(2\pi x)-q}\overset{t=\tanh(2\pi x)}{\cong}$$

$$=-\frac{1}{\pi}\int_0^1\frac{dt}{\frac{1-q}{q}+t^2}=-\frac{1}{\pi}\sqrt{\frac{q}{1-q}}\tan^{-1}\left(\sqrt{\frac{q}{1-q}}\right)$$

$$\frac{d\psi}{dq}=-\frac{1}{2\pi(1-q)\sqrt{q(1-q)}}\tan^{-1}\sqrt{\frac{q}{1-q}}-\frac{1}{2\pi(1-q)}$$

$$\frac{d\psi}{dq}=\frac{1}{2q(1-q)}\psi(q)-\frac{2q+1}{2\pi(1-q)}$$

$$(2q+1)\frac{d\psi}{dq}=\frac{2q+1}{2q(1-q)}\psi(q)-\frac{2q+1}{2\pi(1-q)^2}$$

$$\frac{d^2\psi}{dq^2}=\frac{1}{2q(1-q)}\frac{d^2\psi}{dq^2}+\frac{2q-1}{2q^2(1-q)^2}\psi(q)-\frac{1}{2\pi(1-q)^2}$$

$$2q(q-1)\frac{d^2\psi}{dq^2}=-\frac{d\psi}{dq}-\frac{2q-1}{q(1-q)}\psi(q)+\frac{q}{\pi(1-q)}$$

$$2q(q-1)\frac{d^2\psi}{dq^2}=-\frac{4q-1}{2q(1-q)}\psi(q)+\frac{2q+1}{2\pi(1-q)}$$

$$2q(q-1)\frac{d^2\psi}{dq^2}+(2q+1)\frac{d\psi}{dq}=-\frac{4q-1}{2q(1-q)}\psi(q)+\frac{2q+1}{2\pi(1-q)}+\frac{2q+1}{2q(1-q)}\psi(q)-\frac{2q+1}{2\pi(1-q)}$$

$2q(q-1)\frac{d^2\psi}{dq^2}+(2q+1)\frac{d\psi}{dq}=\frac{\psi(q)}{q}$ **Proved**

$$\psi(q)=\int_{-\infty}^{\infty}\frac{\tanh(4\pi x)-q}{\cosh^2(2\pi x)-q}dx=-\frac{1}{\pi}\sqrt{\frac{q}{1-q}}\tan^{-1}\left(\sqrt{\frac{q}{1-q}}\right)$$

$$2q(q-1)\frac{d^2\psi}{dq^2}+(2q+1)\frac{d\psi}{dq}=\frac{\psi(q)}{q}$$