ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^\infty \left(\sqrt{\tanh(\pi x)}+1\right)\left(\sqrt{\coth(\pi x)}+1\right)e^{-\pi x}dx=\frac{2}{\pi}+\frac{4\sqrt{2}\Gamma\left(\frac{5}{4}\right)^2}{\pi^{\frac{3}{2}}}$$

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$$\Omega=\int_0^\infty \left(\sqrt{\tanh(\pi x)}+1\right)\left(\sqrt{\coth(\pi x)}+1\right)e^{-\pi x}dx=\frac{2}{\pi}+\frac{4\sqrt{2}}{\pi\sqrt{\pi}}\Gamma^2\left(\frac{5}{4}\right)?$$

$$\Omega=\int_0^\infty \left(\sqrt{\tanh(\pi x)}+1\right)\left(\sqrt{\coth(\pi x)}+1\right)e^{-x}dx\overset{x\to\pi x}{=}$$

$$=\frac{1}{\pi}\int_0^\infty\left(\sqrt{\coth(x)}+\sqrt{\tanh(x)}+2\right)e^{-x}dx$$

$$=\frac{2}{\pi}+\frac{1}{\pi}\int_0^\infty\left(\sqrt{\coth(x)}+\sqrt{\tanh(x)}\right)e^{-x}dx$$

Let: $t=\sqrt{\coth(x)}\Rightarrow x=\text{arccoth}(t^2)\Rightarrow dx=\frac{2t}{1-t^4}dt$, also: $e^{-\text{arccoth}(t^2)}=\sqrt{\frac{t^2-1}{t^2+1}}$

$$\Rightarrow\Omega=\frac{2}{\pi}+\frac{1}{\pi}\int_1^\infty\left(t+\frac{1}{t}\right)\sqrt{\frac{t^2-1}{t^2+1}}\frac{2t}{t^4-1}dt=\frac{2}{\pi}+\frac{2}{\pi}\int_1^\infty\frac{1}{\sqrt{t^4-1}}dt\overset{t\to\frac{1}{t}}{\longrightarrow}$$

$$\frac{2}{\pi}+\frac{2}{\pi}\int_0^1\frac{1}{\sqrt{1-t^4}}dt=$$

$$=\frac{2}{\pi}+\frac{1}{2\pi}\int_0^1(t^4)^{\frac{1}{4}-1}(1-t^4)^{\frac{1}{2}-1}d(t^4)=\frac{2}{\pi}+\frac{1}{2\pi}B\left(\frac{1}{4},\frac{1}{2}\right)=\frac{2}{\pi}+\frac{1}{2\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\because\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\pi\sqrt{2}\Rightarrow\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}\text{ and }\Gamma\left(\frac{5}{4}\right)=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Rightarrow\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{4\Gamma\left(\frac{5}{4}\right)}$$

$$\Rightarrow\Omega=\frac{2}{\pi}+\frac{\sqrt{\pi}}{2\pi}\frac{4\Gamma\left(\frac{5}{4}\right)}{\pi\sqrt{2}}4\Gamma\left(\frac{5}{4}\right)=\frac{2}{\pi}+\frac{4\sqrt{2}}{\pi\sqrt{\pi}}\Gamma^2\left(\frac{5}{4}\right)\text{, hence proved.}$$