ROMANIAN MATHEMATICAL MAGAZINE

Find

$$\int_0^1 \frac{\tan^{-1}(x)\tanh^{-1}(x)}{1+x^2}dx$$

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$$\lambda = \int_0^1 \frac{\tan^{-1}(x)\tanh^{-1}(x)}{1+x^2}dx = \frac{1}{2}\int_0^1 \frac{\tan^{-1}(x)\log\left(\frac{1+x}{1-x}\right)}{1+x^2}dx$$

$$let\left\{\tan^{-1}(x) = t,\ \ dt = \frac{dx}{1+x^2}, t\left[\frac{\pi}{4};0\right]\right\}$$

$$\lambda = \frac{1}{2}\int_0^{\frac{\pi}{4}} t\log\left(\frac{1+\tan(t)}{1-\tan(t)}\right)dt = \frac{1}{2}\int_0^{\frac{\pi}{4}} t\log\left(\tan\left(t+\frac{\pi}{4}\right)\right)dt =$$

$$let:\left\{\theta = \left(t+\frac{\pi}{4}\right), d\theta = dt, t = \theta - \frac{\pi}{4}\ \ \theta\left[\frac{\pi}{2};\frac{\pi}{4}\right]\right\}$$

$$\lambda = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\left(\theta-\frac{\pi}{4}\right)\log(\tan(\theta))d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\theta\log(\tan(\theta))d\theta - \frac{\pi}{8}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\log(\tan(\theta))d\theta = \frac{1}{2}\lambda_1 - \frac{\pi}{8}\lambda_2$$

$$Note: \{\ Fourier\ series\ of\ log\big(tan(\theta)\big) = -2\sum_{n=0}^{\infty}\frac{\cos(4n\theta+2\theta)}{2n+1}\}$$

$$\lambda_1 = -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\theta\cos\big(\theta(4n+2)\big)d\theta = -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{\theta\, s\, in\big(\theta(4n+2)\big)}{4n+2} + \frac{\cos\big(\theta(4n+2)\big)}{(4n+2)^2}\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}$$

$$= -2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{2\sin(\pi n)}{4(2n+1)^2} - \frac{\cos(2\pi n)}{4(2n+1)^2} - \frac{\pi\sin(\pi n)\cos(\pi n)}{2(2n+1)} - \frac{\pi\cos(\pi n)}{8(2n+1)}\right]$$

$$Notes: \{\sin(\pi n) = 0, \cos(2\pi n) = 1, \cos(\pi n) = (-1)^n, n\epsilon N\cup\{0\}\}$$

$$\lambda_1 = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{1}{4(2n+1)^2} + \frac{\pi(-1)^n}{8(2n+1)}\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} + \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$Notes:$$

$$\left\{Dirichlet\ beta\ function: \beta(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z}, \beta(2) = G\ (Catalan's constant)\right\}$$

$$\left\{Rieman's\ zeta\ function: \zeta(z) = \frac{1}{1-2^{-z}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^z}\right\}$$

$$\lambda_1=\frac{1}{2}(1-2^{-3})\zeta(3)+\frac{\pi G}{4}=\frac{7}{16}\zeta(3)+\frac{\pi G}{4}$$

$$\lambda_2=\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\log(\tan(\theta))d\theta=-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\cos(\theta(4n+2))d\theta=$$

$$=-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{\sin(\theta(4n+2))}{2(2n+1)}\right]_{\frac{\pi}{2}}^{\frac{\pi}{4}}=\sum_{n=0}^{\infty}\frac{1}{2n+1}\left[\frac{\sin(2\pi n)}{2n+1}+\frac{\cos(\pi n)}{2n+1}\right]=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G$$

$$\int_0^1\frac{\tan^{-1}(x)\tanh^{-1}(x)}{1+x^2}dx=\frac{1}{2}\lambda_1-\frac{\pi}{8}\lambda_2=\frac{7}{32}\zeta(3)+\frac{\pi G}{8}-\frac{\pi G}{8}=\frac{7}{32}\zeta(3)$$