ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \le b$, then :

$$4\int_a^b \tanh x\,dx \ge \cos 2a - \cos 2b$$

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$$4\int_a^b \tanh x\,dx = 4\int_a^b \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = 4\int_a^b \frac{(e^x + e^{-x})'}{e^x + e^{-x}}dx$$

$$= 4\ln\left(e^b + e^{-b}\right) - 4\ln(e^a + e^{-a}) \overset{?}{\ge} \cos 2a - \cos 2b$$

$$\Leftrightarrow 4\ln\left(e^b + e^{-b}\right) + \cos 2b \overset{?}{\ge} 4\ln(e^a + e^{-a}) + \cos 2a$$

$$\Leftrightarrow f(b) - f(a) \overset{?}{\ge} 0\ (f(x) = 4\ln(e^x + e^{-x}) + \cos 2x\ \forall\, x > 0) \overset{\text{via MVT}}{\Leftrightarrow}$$

$$(b-a)f'(c) \overset{?}{\ge} 0\ (a \le c \le b) \Leftrightarrow f'(c) \overset{?}{\ge} 0\ (\because b \ge a) \Leftrightarrow 4.\frac{e^c - e^{-c}}{e^c + e^{-c}} - 2\sin 2c \overset{?}{\ge} 0$$

$$\Leftrightarrow 4.\frac{e^{2c} + 1 - 2}{e^{2c} + 1} - 2\sin 2c \overset{?}{\ge} 0 \Leftrightarrow \boxed{2 - \sin 2c - \frac{4}{e^{2c} + 1} \overset{?}{\ge} 0} \to (1)$$

$$\text{Let } F(y) = \sin y - y + \frac{y^3}{6} - \frac{y^5}{120}\ \forall\, y \in [0,\infty)$$

$$\therefore F'(y) = \cos y - \frac{y^4}{24} + \frac{y^2}{2} - 1 \text{ and } F''(y) = -\left(\sin y - \left(y - \frac{y^3}{6}\right)\right)$$

$$\text{Let } P(y) = \sin y - y + \frac{y^3}{6}\ \forall\, y \in [0,\infty) \therefore P'(y) = \cos y + \frac{y^2}{2} - 1 \text{ and}$$

$$P''(y) = y - \sin y \ge 0 \Rightarrow P'(y) \text{ is } \uparrow \text{ on } [0,\infty) \Rightarrow P'(y) \ge P'(0) = 0 \Rightarrow P(y) \text{ is } \uparrow$$

$$\text{on } [0,\infty) \Rightarrow P(y) \ge P(0) = 0 \Rightarrow \sin y \ge y - \frac{y^3}{6}\ \forall\, y \in [0,\infty)$$

$$\Rightarrow F''(y) \le 0 \Rightarrow F'(y) \text{ is } \downarrow \text{ on } [0,\infty) \Rightarrow F'(y) \le F'(0) = 0 \Rightarrow F(y) \text{ is } \downarrow \text{ on } [0,\infty)$$

$$\Rightarrow F(y) \le F(0) = 0 \Rightarrow \sin y \le y - \frac{y^3}{6} + \frac{y^5}{120}\ \forall\, y \in [0,\infty)$$

$$\therefore \forall\, y \in (0,\infty), \sin y < y - \frac{y^3}{6} + \frac{y^5}{120} \to (i)$$

$$\text{Let } h(y) = e^y - 1 - y - \frac{y^2}{2}\ \forall\, y \in [0,\infty) \therefore h'(y) = e^y - 1 - y \ge 0 \Rightarrow h(y) \text{ is } \uparrow$$

$$\text{on } [0,\infty) \Rightarrow h(y) \ge h(0) = 0 \Rightarrow \forall\, y \in (0,\infty), e^y > 1 + y + \frac{y^2}{2} \to (ii)$$

$$\boxed{\text{Case 1}}\ c \ge 1 \text{ and we have : } 2 - \sin 2c - \frac{4}{e^{2c} + 1} > 1 - \sin 2c + 1 - \frac{4}{2c + 1 + 1}$$

$$= 1 - \sin 2c + 1 - \frac{2}{c + 1} = (1 - \sin 2c) + \frac{c - 1}{c + 1} \ge 0 \Rightarrow (1) \text{ is true}$$

$\boxed{\mathit{Case}\ 2}$ $0 < c < 1$ *and via* (i), (ii), $2 - \sin 2c - \frac{4}{e^{2c} + 1} >$

$$2 - \left(2c - \frac{8c^3}{6} + \frac{32c^5}{120}\right) - \frac{4}{\left(1 + c + \frac{c^2}{2}\right)^2 + 1}$$

$$= 2\left(\frac{15 - 15c + 20c^3 - 4c^5}{15} - \frac{8}{4(1 + c)^2 + c^4 + 4c^2(1 + c) + 4}\right)$$

$$= \frac{-2c^3\left(4c^6 + 16c^5 + 12c^4 - 48c^3 - 113c^2 - 115c - 100\right)}{15(4(1 + c)^2 + c^4 + 4c^2(1 + c) + 4)}$$

$$= \frac{-2c^3\left(4\left(c^6 - 1\right) + 16\left(c^5 - 1\right) + 12\left(c^4 - 1\right) - 48c^3 - 113c^2 - 115c - 68\right)}{15(4(1 + c)^2 + c^4 + 4c^2(1 + c) + 4)} > 0$$

$\because 0 < c < 1 \Rightarrow 4\left(c^6 - 1\right) + 16\left(c^5 - 1\right) + 12\left(c^4 - 1\right) - 48c^3 - 113c^2 - 115c - 68$ $< 0 \Rightarrow$ **(1) is true** $\therefore$ combining both *cases*, (1) is true $\forall\, c \in (0, \infty)$

$$\therefore 4\int_a^b \tanh x\, dx \geq \cos 2a - \cos 2b \text{ whenever } 0 < a \leq b \text{ (QED)}$$