ROMANIAN MATHEMATICAL MAGAZINE

If $0 < a \le b$ then:

$$e^a + e^{-a} + 2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{x^2+1}\right)}dx \le e^b + e^{-b}$$

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Solution by Hikmat Mammadov-Azerbaijan

$$0 < a \le b; c^a + c^{-a} + 2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{x^2+1}\right)}dx \le c^b + c^{-b}$$

Since sinh convex on $\mathbb{R}_+ \Rightarrow \forall t \in \mathbb{R}_+, \sinh(t) \ge t \ge 0$

$$\forall t \in \mathbb{R}_+, \sinh^2(t) \ge t^2, \qquad \forall t \in \mathbb{R}_+, 1+\sinh^2(t) \ge 1+t^2$$

$$\forall t \in \mathbb{R}_+, \cosh^2(t) \ge 1+t^2$$

$$\forall t \in \mathbb{R}_+, \sqrt{\cosh(t)} \ge \sqrt[4]{1+t^2}$$

So $\Rightarrow \forall t \in \mathbb{R}_+, \frac{\sqrt{\cosh(t)}}{\sqrt[4]{1+t^2}} \ge 1$

The Cauchy – Schwarz inequality gives $\Rightarrow$

$$\forall x \in \mathbb{R}_+, \int_0^x \cosh(t)\,dt \int_0^x \frac{dt}{\sqrt{1+t^2}} \ge \left(\int_0^x \frac{\sqrt{\cosh(t)}}{\sqrt[4]{1+t^2}}\right)^2$$

$$\forall x \in \mathbb{R}_+, \int \cosh(t)\,dt \int_0^x \frac{dt}{\sqrt{1+t^2}} \ge \left(\int_0^x dt\right)^2$$

i.e: $\forall x \in \mathbb{R}_+, \sinh(x)\,\text{arcsinh}(x) \ge x^2$

So $\Rightarrow \forall x \in [a;b], \frac{x^2}{\text{arcsinh}(x)} \le \sinh(x)$

$$\int_a^b \frac{x^2}{\text{arcsinh}(x)}dx \le \int_a^b \sinh(x)\,dx$$

i.e: $\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le \cosh(a) - \cosh(b)$

$$2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le e^b + e^{-b} - e^a - e^{-a}$$

Finally $\Rightarrow c^a + c^{-a} + 2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le c^b + c^{-b}$