ROMANIAN MATHEMATICAL MAGAZINE

Suppose:

$$f(x)=\int_0^{\infty}\ln(1+e^{-2t})\,dt$$

Prove without any software:

$$\frac{3}{8}\le f(x)\le\frac{1}{4}(1+\ln(2))$$

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$$*\, I=\int_0^{\infty}\ln(1+e^{-2t})\,dt, u=e^{-2t}\Rightarrow -2t=\ln(u)\Rightarrow dt=-\frac{1}{2}\frac{du}{u}\Rightarrow$$

$$\Rightarrow I=\frac{1}{2}\int_0^{1}\frac{\ln(1+u)}{u}du=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 u^{n-1}\,du=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{24}$$

* We need to prove: $\frac{3}{8}\le\frac{\pi^2}{24}\le\frac{1}{4}(1+\ln(2))$

a) $\frac{3}{8}\le\frac{\pi^2}{24}\Leftrightarrow \pi^2\ge 9$

Since the series expansion of $\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}\dots\Rightarrow$

$\Rightarrow \sin(x)\le x\ \ \forall x\in\mathbb{R}\Rightarrow \sin\left(\frac{\pi}{6}\right)=\frac{1}{2}\le\frac{\pi}{6}\Rightarrow\pi>3\Leftrightarrow\pi^2\ge 9$, done.

b) $\frac{\pi^2}{24}\le\frac{1}{4}(1+\ln(2))\Leftrightarrow\pi^2\le 6+6\ln(2)$. We will show that: $\pi^2\le 10$

Indeed, we have: $0<\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx=\frac{22}{7}-\pi\Rightarrow\pi<\frac{22}{7}\Rightarrow$

$\Rightarrow\pi^2<\frac{484}{49}\le 10$, true since: $10=\frac{490}{49}$

Now, we will show that: $10\le 6+6\ln(2)\Leftrightarrow\ln(2)\ge\frac{2}{3}$,

Since the series expansion of: $\ln\left(\frac{1+x}{1-x}\right)=2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\cdots\Rightarrow$

$\Rightarrow\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)=\frac{2}{3}+\frac{2}{3}\left(\frac{1}{3}\right)^3+\cdots\Rightarrow\ln(2)=\frac{2}{3}+\frac{2}{3}\left(\frac{1}{3}\right)^3+\cdots\Rightarrow\ln(2)\ge\frac{2}{3}$, true

$\Rightarrow\pi^2\le 10\le 6+6\ln(2)$, done. Combine all results we conclude that:

$$\frac{3}{8}\le\frac{\pi^2}{24}\le\frac{1}{4}(1+\ln(2))$$