ROMANIAN MATHEMATICAL MAGAZINE

Let be $a_1 = a \in \mathbb{R}$, $a \neq 2$, $a_2 = \frac{a}{a-2}$, and: $a_{n+1} = \frac{a_n^2}{a_n^2 - 2a_n + 2}$, $\forall n \geq 2$

Prove that:

$$\sum_{k=1}^{n} 2^{k-1} a_k = \prod_{k=1}^{n} a_k.$$

Determine a_n and compute:

$$\lim_{n\to\infty} a_n$$

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Let us prove the following issue by mathematical induction:

$$E(n) \ : \quad \sum_{k=1}^{n} 2^{k-1} a_k = \prod_{k=1}^{n} a_k$$

At $= 1$: $a_1 = a_1$, so the issue is valid for $n = 1$.

$$\text{At} = 2 : a_1 + 2a_2 \overset{?}{=} a_1.a_2$$

Firstly, $l_1 := a_1 + 2a_2 = a_1 + \frac{2a_1}{a_1-2} = \frac{a_1^2 - 2a_1 + 2a_1}{a_1 - 2} = \frac{a_1^2}{a_1-2} = a_1.\frac{a_1}{a_1-2} = a_1.a_2 := l_2$

Now, we assume that the issue is valid at n=p

$$E(p) \ : \quad \sum_{k=1}^{p} 2^{k-1} a_k = \prod_{k=1}^{p} a_k$$

And let's prove its validity at n=p+1:

$$E(p+1) \ : \quad \sum_{k=1}^{p+1} 2^{k-1} a_k = \prod_{k=1}^{p+1} a_k$$

$$l_1 := \sum_{k=1}^{p+1} 2^{k-1} a_k = \sum_{k=1}^{p} 2^{k-1} a_k + 2^p a_{p+1} = \prod_{k=1}^{p} a_k + \frac{2^p a_p^2}{a_p^2 - 2a_p + 2}$$

$$= \frac{a_p^2 \prod_{k=1}^{p} a_k - 2a_p \prod_{k=1}^{p} a_k + 2 \prod_{k=1}^{p} a_k + 2^p a_p^2}{a_p^2 - 2a_p + 2}$$

Let's look at:

$$\prod_{k=1}^{p-1} a_k + 2^{p-1} a_p = \sum_{k=1}^{p-1} 2^{k-1} a_k + 2^{p-1} a_p = \sum_{k=1}^{p} 2^{k-1} a_k = \prod_{k=1}^{p} a_k$$

$$\Rightarrow -\prod_{k=1}^{p} a_k + \prod_{k=1}^{p-1} a_k + 2^{p-1} a_p = -\prod_{k=1}^{p} a_k + \prod_{k=1}^{p} a_k = 0$$

$$\Rightarrow -\prod_{k=1}^{p} a_k + \prod_{k=1}^{p-1} a_k + 2^{p-1} a_p = 0$$

We multiply both sides by $2a_p$:

$$\Rightarrow -2a_p \prod_{k=1}^{p} a_k + 2a_p \prod_{k=1}^{p-1} a_k + 2^p a_p^2 = 2a_p(0) = 0$$

$$\Rightarrow \quad -2a_p\prod_{k=1}^{p} a_k + 2\prod_{k=1}^{p} a_k + 2^p a_p^2 = 0$$

Hence,

$$l_1 = \frac{a_p^2\prod_{k=1}^{p} a_k - 2a_p\prod_{k=1}^{p} a_k + 2\prod_{k=1}^{p} a_k + 2^p a_p^2}{a_p^2 - 2a_p + 2} = \prod_{k=1}^{p} a_k \frac{a_p^2}{a_p^2 - 2a_p + 2} =$$

$$= a^{p+1}\prod_{k=1}^{p} a_k = \prod_{k=1}^{p+1} a_k \coloneqq l_2$$

So, $E(p+1)$ is valid and the issue is valid for any $n \ge 1$.

Now, let's consider a function f defined on $\mathbb{R}$ by :

$$f : x \mapsto f(x) \coloneqq \frac{x^2}{x^2 - 2x + 2}$$

This function is differentiable on $\mathbb{R}$ and :

$$f'(x) = -\frac{2x(x-2)}{(x^2 - 2x + 2)^2}$$

So, f is decreasing on $I_1 \coloneqq]-\infty, 0]$ and $0 = f(0) \le f(x) < \lim_{x\to-\infty} f(x) = 1$ on I_1.

On $I_2 \coloneqq [0,2]$ the function is increasing and $0 = f(0) \le f(x) \le f(2) = 2$ on I_2.

On $I_3 \coloneqq [2, +\infty[$ the function is decreasing and $1 = \lim_{x\to+\infty} f(x) < f(x) \le 2$ on I_3

Hence, For all $\in \mathbb{R}$, $0 \le f(x) \le 2$.

Since $f([0,2]) = [0,2]$ and $f(x) \in [0,2]$ for any $x \in \mathbb{R}$ then $(a_n)_{n\ge1}$ is bounded and $a_n \in [0,2]$ when $n \ge 3$.

On the other hand,

$$a_{n+1} - a_n = -\frac{a_n(a_n - 2)(a_n - 1)}{a_n^2 - 2a_n + 2} \quad , \qquad n \ge 2$$

$if\ a_1 = a > 2 \ \Rightarrow\ a_2 = \frac{a}{a-2} > 1 \ \Rightarrow\ a_3 - a_2 > 0$ and while $0 \le a_n \le 2$, $n \ge 3$ then a_3 stays in the interval $]1,2]$, hence $a_4 - a_3 > 0$, so a_4 stays in the same interval, hence the same is true for $a_5, a_6, \dots$ .We deduce that $(a_n)_{n\ge2}$ is increasing.

So, $(a_n)_{n\ge1}$ is increasing, then it's convergent to a number such as x

$$x = \frac{x^2}{x^2 - 2x + 2} \iff x(x^2 - 3x + 2) = 0 \iff x(x-2)(x-1) = 0$$

Since the sequence is increasing then $x = 2$ and $\lim_{n\to\infty} a_n = 2$.

$if\ a_1 = a < 2 \ \Rightarrow\ a_2 = \frac{a}{a-2} < 1 \ \Rightarrow\ a_3 - a_2 < 0$ and while $0 \le a_n \le 2$, $n \ge 3$ then a_3 stays in the interval $[0,1[$, hence $a_4 - a_3 < 0$, so a_4 stays in the same interval too, hence the same is true for $a_5, a_6, \dots$ .We deduce that $(a_n)_{n\ge2}$ is decreasing.

So, $(a_n)_{n\ge1}$ is decreasing, then it's convergent to a number such as x

$$x = \frac{x^2}{x^2 - 2x + 2} \iff x(x^2 - 3x + 2) = 0 \iff x(x-2)(x-1) = 0$$

Since the sequence is decreasing then $x = 0$ and $\lim_{n\to\infty} a_n = 0$.