ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\lim_{n\to\infty}\frac{n+2}{2^n}\sum_{k=0}^{n}\frac{(-1)^k\cdot 2^{n-k}}{k+1}\cdot\binom{n}{k}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Pham Duc Nam-Vietnam

$$(-x+2)^n=\sum_{k=0}^{n}\binom{n}{k}(-x)^k 2^{n-k}\Rightarrow\int(-x+2)^n\,dx=\int\sum_{k=0}^{n}\binom{n}{k}(-x)^k2^{n-k}dx$$

$$\Leftrightarrow -\frac{(2-x)^{n+1}}{n+1}+C=\sum_{k=0}^{n}\binom{n}{k}\frac{x(-x)^k}{k+1}2^{n-k},x=0\Rightarrow-\frac{2^{n+1}}{n+1}+C=0\Rightarrow C=\frac{2^{n+1}}{n+1}$$

$$\Rightarrow\sum_{k=0}^{n}\binom{n}{k}\frac{x(-x)^k}{k+1}2^{n-k}=-\frac{(2-x)^{n+1}}{n+1}+\frac{2^{n+1}}{n+1}$$

Let: $x=1\Rightarrow\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}2^{n-k}=-\frac{1}{n+1}+\frac{2^{n+1}}{n+1}=\frac{2^{n-1}-1}{n+1}$

$$\Rightarrow\lim_{n\to\infty}\frac{n+2}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}2^{n-k}=\lim_{n\to\infty}\frac{n+2}{2^n}\frac{2^{n-1}-1}{n+1}=$$

$$=\lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{2^{n-1}-1}{2^{n-1}}\cdot\frac{1}{2}=\frac{1}{2}$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\Omega=\lim_{n\to\infty}\frac{n+2}{2^n}\sum_{k=0}^{n}\frac{(-1)^k\cdot2^{n-k}}{k+1}\cdot\binom{n}{k}$$

$$\therefore\ (1-x)^n=\sum_{k=0}^{n}(-1)^k\cdot x^k\cdot\binom{n}{k}$$

$$\Rightarrow\int_0^t\sum_{k=0}^{n}(-1)^k\cdot x^k\cdot\binom{n}{k}=\frac{1}{n+1}-\frac{(1-x)^{n+1}}{n+1}$$

$$\Rightarrow\sum_{k=0}^{n}\frac{(-1)^k\cdot x^{k+1}}{k+1}\binom{n}{k}=\frac{1}{n+1}\cdot\left(1-(1-x)^{n+1}\right)$$

$$\Rightarrow \frac{n+2}{2^n}\sum_{k=0}^{n}\frac{(-1)^k\cdot 2^{n-k}}{k+1}\cdot\binom{n}{k}=(n+2)\sum_{k=0}^{n}\frac{(-1)^k\cdot\left(\frac{1}{2}\right)^k}{k+1}\cdot\binom{n}{k}$$

$$=2\cdot(n+2)\sum_{k=0}^{n}\frac{(-1)^k\cdot\left(\frac{1}{2}\right)^{k+1}}{k+1}\cdot\binom{n}{k}=2\cdot(n+2)\cdot\frac{1}{n+1}\left(1-\left(1-\frac{1}{2}\right)^{n+1}\right)=2$$

$$\Rightarrow \Omega=2$$