ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=2}^{n}\sqrt{\frac{(k-1)k}{2k-1+2\sqrt{(k-1)k}}}\cdot\left(\sum_{k=1}^{n}\sqrt{k}\right)^{-1}$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=2}^{n}\sqrt{\frac{(k-1)k}{2k-1+2\sqrt{(k-1)k}}}\cdot\left(\sum_{k=1}^{n}\sqrt{k}\right)^{-1}$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=2}^{n}\frac{\sqrt{k\cdot(k-1)}}{\sqrt{\left(\sqrt{k-1}+\sqrt{k}\right)^{2}}}\cdot\left(\sum_{k=1}^{n}\sqrt{k}\right)^{-1}$$

$$S_1=\sum_{k=2}^{n}\frac{\sqrt{k\cdot(k-1)}}{\sqrt{k-1}+k}=\sum_{k=1}^{n}\frac{\sqrt{k\cdot(k-1)}}{\sqrt{k-1}+\sqrt{k}}=\sum_{k=2}^{n}\left(k\cdot\sqrt{k-1}-(k-1)\cdot\sqrt{k}\right)$$

$$=\sum_{k=2}^{n}\left((k-1)\cdot\sqrt{k-1}+\sqrt{k-1}-k\cdot\sqrt{k}+\sqrt{k}\right)$$

$$=\sum_{k=2}^{n}\left((k-1)\cdot\sqrt{k-1}-k\cdot\sqrt{k}\right)+\sum_{k=2}^{n}\left(\sqrt{k}+\sqrt{k-1}\right)$$

$$=-n\cdot\sqrt{n}+1+\left(2\sum_{k=2}^{n}\sqrt{k}-\sqrt{n-1}+1\right)\Rightarrow\sqrt{k-1}\le\int_{k-1}^{k}\sqrt{x}\le\sqrt{k}$$

$$\Rightarrow\sum_{k=2}^{n}\sqrt{k-1}\le\int_{1}^{n}\sqrt{x}\le\sum_{k=2}^{n}\sqrt{k}=S\Rightarrow S-\sqrt{n}+1\le\frac{2}{3}\cdot n\sqrt{n}-\frac{2}{3}\le S\Rightarrow$$

$$\Rightarrow S\sim\frac{2}{3}\cdot n\sqrt{n}$$

$$\Rightarrow\Omega=\frac{\frac{1}{n}\cdot\left(-n\sqrt{n}+2+2S-2\sqrt{n-1}+1\right)}{S}\Rightarrow$$

$$\Rightarrow\Omega\sim\frac{1}{n}\cdot\left(\frac{1}{3}\cdot n\sqrt{n}\right)\cdot\left(\frac{2}{3}\cdot n\sqrt{n}\right)^{-1}=0\Rightarrow\Omega=0$$