ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n+1]{(n+1)!}}\cdot\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}}}\right)$$

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Solution by Adrian Popa – Romania

First, we will calculate:

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} \overset{C.D.}{=} \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} =$$

$$= \lim_{n\to\infty}\frac{(n+1)^n\cdot(n+1)\cdot n!}{n!\,(n+1)n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \Rightarrow$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}} = e$$

$$\Omega = \lim_{n\to\infty}\frac{1}{\sqrt[n+1]{(n+1)!}}\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}}} = \lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}}\cdot\frac{1}{n+1}\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}}}$$

$$= \lim_{n\to\infty} e\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}(n+1)^5}} = e\cdot\lim_{n\to\infty}\sqrt[5]{\frac{H_n}{H_{n+1}\cdot n(n+1)^4}} = 0$$