ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega(x)=\frac{\sum_{k=1}^{\infty}\left(\frac{\sin k}{k}\right)x^k}{\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}},\qquad x\in[-1,1]$$

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Solution by Pham Duc Nam-Vietnam

$$\Omega(x)=\frac{\sum_{k=1}^{\infty}\frac{\sin(k)}{k}x^k}{\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}},x\in[-1,1]$$

$$*\sum_{k=1}^{\infty}\frac{\sin(k)}{k}x^k=\Im\sum_{k=1}^{\infty}\frac{e^{ik}}{k}x^k=\Im\sum_{k=1}^{\infty}\frac{(xe^{i})^k}{k}=\Im(-\ln(1-xe^{i}))$$

$$=\Im(-\ln(1-x\cos(1)-ix\sin(1)))=\arctan\left(\frac{x\sin(1)}{1-x\cos(1)}\right)$$

$$*\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}=\prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2}=\prod_{k=1}^{\infty}\left(1-\left(\frac{1}{2k}\right)^2\right)$$

$$\frac{\sin(\pi z)}{\pi z}=\prod_{k=1}^{\infty}\left(1-\left(\frac{z}{k}\right)^2\right),\text{ let: } z=\frac{1}{2}\Rightarrow\prod_{k=1}^{\infty}\left(1-\left(\frac{1}{2k}\right)^2\right)=\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}=\frac{2}{\pi}$$

$$\Rightarrow\Omega(x)=\frac{\sum_{k=1}^{\infty}\frac{\sin(k)}{k}x^k}{\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}}=\frac{\pi}{2}\arctan\left(\frac{x\sin(1)}{1-x\cos(1)}\right)$$