ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega(a,b,c)=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)(a^x+b^x+c^x)^{\frac{1}{x}}}{3^{\frac{1}{x}}\left((1+x)^{\frac{1}{x}}-e\right)},a,b,c>0$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution 1 by Pham Duc Nam-Vietnam

$$\Omega(a,b,c)=\lim_{x\to 0}\frac{x\left(x\sin\left(\frac{1}{x}\right)+1\right)}{(1+x)^{\frac{1}{x}}-e}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}},a,b,c>0$$

$$=\frac{1}{e}\lim_{x\to 0}\frac{x\left(x\sin\left(\frac{1}{x}\right)+1\right)^{\frac{1}{x\sin\left(\frac{1}{x}\right)}x\sin\left(\frac{1}{x}\right)}}{e^{\frac{1}{x}\ln(1+x)-1}-1}\left(x\frac{\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}+\frac{3}{x}}{3}\right)^{\frac{1}{x}}=$$

$$=\frac{1}{e}\lim_{x\to 0}\frac{xe^{x\sin\left(\frac{1}{x}\right)}}{\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{\frac{1}{x}\ln(1+x)-1}}\frac{1}{\frac{1}{x}\ln(1+x)-1}\left(x\frac{\ln(abc)+\frac{3}{x}}{3}\right)^{\frac{1}{x}}$$

$$=\frac{1}{e}\lim_{x\to 0}e^{x\sin\left(\frac{1}{x}\right)}\frac{x}{\frac{1}{x}\ln(1+x)-1}\left(1+\frac{1}{3}x\ln(abc)\right)^{\frac{1}{x}}=$$

$$=\frac{1}{e}\lim_{x\to 0}\frac{x^2}{\ln(1+x)-x}\left(1+\frac{1}{3}x\ln(abc)\right)^{\frac{3}{x\ln(abc)}\frac{1}{3}\ln(abc)}$$

$$=\sqrt[3]{abc}\frac{1}{e}\lim_{x\to 0}\frac{x^2}{\ln(1+x)-x}\left(\frac{0}{0}\right)\overset{L'H}{\to}-\sqrt[3]{abc}\frac{1}{e}\lim_{x\to 0}\frac{2x}{\frac{x}{1+x}}=$$

$$=-2\sqrt[3]{abc}\frac{1}{e}\lim_{x\to 0}(1+x)=-\frac{2\sqrt[3]{abc}}{e}$$

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Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)(a^x+b^x+c^x)^{\frac{1}{x}}}{3^{\frac{1}{x}}\left((1+x)^{\frac{1}{x}}-e\right)}=\underbrace{\left(\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}-1\right)}{(1+x)^{\frac{1}{x}}-e}\right)}_{\frac{0}{0}}\underbrace{\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}}_{1^{\infty}}=$$

$$=\left(-\frac{2}{e}\right)\left(\sqrt[3]{abc}\right)=-\frac{2\sqrt[3]{abc}}{e}$$

where

$$\text{(i) }\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{(1+x)^{\frac{1}{x}}-e}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e^{\frac{1}{x}\ln(1+x)}-e}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e^{\frac{1}{x}\left(x-\frac{1}{2}x^2+O(x^3)\right)}-e}=$$

$$=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e^{1-\frac{1}{2}x+O(x^2)}=e}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e\left(e^{-\frac{1}{2}x+O(x^2)}-1\right)}$$

$$=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e\left(\left(1-\frac{1}{2}x+O(x^2)-1\right)\right)}=\lim_{x\to 0}\frac{x\left(x\sin\frac{1}{x}+1\right)}{e\left(-\frac{1}{2}x+O(x^2)\right)}=$$

$$=\lim_{x\to 0}\frac{x\sin\frac{1}{x}+1}{e\left(-\frac{1}{2}+O(x)\right)}=\frac{0+1}{e\left(-\frac{1}{2}\right)}=-\frac{2}{e}$$

$$\text{(ii) }\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}=\exp\left\{\underbrace{\lim_{x\to 0}\frac{\ln(a^x+b^x+c^x)-\ln 3}{x}}_{\frac{0}{0}}\right\}=\exp\left\{\underbrace{\lim_{x\to 0}\frac{\frac{a^x\ln a+b^x\ln b+c^x\ln c}{a^x+b^x+c^x}}{1}}_{L'Hopital\ Rule}\right\}$$

$$=\exp\left\{\frac{\ln a+\ln b+\ln c}{3}\right\}=e^{\ln(abc)^{\frac{1}{3}}}=\sqrt[3]{abc}$$