ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \lim_{n\to\infty}\left(logn + \lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

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Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \lim_{n\to\infty}\left(\log(n) + \lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right) = \lim_{n\to 0}(\log(n) + f(n))$$

$$f(n) = \lim_{x\to 0}\frac{1+x^2-(1+x^2)^{H_n}}{x^2(1+x^2)} \quad \{1+x^2 = t\ , \qquad x^2 = t-1\}$$

$$f(n) = \lim_{t\to 1}\frac{t-t^{H_n}}{t^2-t} = \lim_{x\to 1}\frac{\frac{\partial}{\partial t}(t-t^{H_n})}{\frac{\partial}{\partial t}(t^2-t)} = \lim_{t\to 1}\frac{1-H_n t^{H_n-1}}{2t-1} = 1-H_n$$

$$\Omega = \lim_{n\to 0}(\log(n) - H_n + 1) = \lim_{n\to\infty}\left(\log(n) - \ln(n) - \gamma - \frac{1}{2n} + \xi_n + 1\right) = 1-\gamma$$

$$\left\{0 \le \xi_n \le \frac{1}{8n^2}\ \ n\to\infty\ \ \xi_n \to 0\right\}$$