ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\lim_{n\to\infty}\left(\log n+\lim_{x\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)$$

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Solution by Amin Hajiyev – Azerbaijan

$$\Omega=\lim_{n\to\infty}\left(\log(n)+\lim_{n\to 0}\frac{1-(1+x^2)^{H_n-1}}{x^2}\right)=\lim_{n\to 0}\big(\log(n)+f(n)\big)$$

$$f(n)=\lim_{n\to 0}\frac{1+x^2-(1+x^2)^{H_n}}{x^2(1+x^2)}\ \{1+x^2=t\ \ x^2=t-1\}$$

$$f(n)=\lim_{t\to 1}\frac{t-t^{H_n}}{t^2-t}=\lim_{x\to 1}\frac{\frac{\partial}{\partial t}(t-t^{H_n})}{\frac{\partial}{\partial t}(t^2-t)}=\lim_{t\to 1}\frac{1-H_nt^{H_n-1}}{2t-1}=1-H_n$$

$$\Omega=\lim_{n\to\infty}(\log(n)-H_n+1)=\lim_{n\to\infty}\left(\log(n)-\ln(n)-\gamma-\frac{1}{2n}+\xi_n+1\right)=1-\gamma$$

$$\left\{0\le\xi_n\le\frac{1}{8n^2}\ \ n\to\infty\quad \xi_n\to 0\right\}$$