ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\lim_{n\to\infty}\log n\cdot\left(\sum_{k=2}^{n}\frac{\log k}{k}\right)^{\frac{1}{n}}\cdot\left(\int_0^n\frac{|\sin x|}{x}\right)^{-1}$$

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Euler – Maclaurin summation gives us

$$\sum_{k=2}^{n}\frac{\log k}{k}=\sum_{k=1}^{n}\frac{\log k}{k}\sim\frac{\log^2 n}{2}$$

as $n\to\infty$, so that

$$\lim_{n\to\infty}\left(\sum_{k=2}^{n}\frac{\log k}{k}\right)^{\frac{1}{n}}=1.$$

We also have $\int_1^n\frac{|\sin x|}{x}dx\sim\frac{2}{\pi}\log n$ as $n\to\infty$ (proved below), and it follows that

$$\lim_{n\to\infty}\log n\left(\sum_{k=2}^{n}\frac{\log k}{k}\right)^{\frac{1}{n}}\left(\int_1^n\frac{|\sin x|}{x}dx\right)^{-1}=\frac{\pi}{2}$$

Let $m=\left\lfloor\frac{n}{\pi}\right\rfloor$ so that $m\pi\le n<m\pi+\pi$. We have

$$\int_1^n\frac{|\sin x|}{x}dx=\int_1^\pi\frac{|\sin x|}{x}dx+\sum_{k=2}^{m}\int_{(k-1)\pi}^{k\pi}\frac{|\sin x|}{x}dx+\int_{m\pi}^{n}\frac{|\sin x|}{x}dx\qquad(1)$$

Using $\int_{j\pi}^{j\pi+\pi}|\sin x|\,dx=2$, we can find the following upper and lower bounds for the terms on the RHS of (1):

$$0\le\int_1^\pi\frac{|\sin x|}{x}dx\le\log\pi,\ \frac{2}{\pi}\sum_{k=2}^{m}\frac{1}{k}\le\sum_{k=2}^{m}\int_{(k-1)\pi}^{k\pi}\frac{|\sin x|}{x}dx\le\frac{2}{\pi}\sum_{k=2}^{m}\frac{1}{k-1}\qquad(2)$$

$$0\le\int_{m\pi}^{n}\frac{|\sin x|}{x}dx\le\int_{m\pi}^{m\pi+\pi}dx\le\frac{2}{m\pi+\pi}$$

Dividing both sides of (1) by $\log n$ and applying the bounds in (2), we get

$$\frac{2}{\pi\log n}\sum_{k=2}^{m}\frac{1}{k}\le\frac{1}{\log n}\int_1^n\frac{|\sin x|}{x}dx\le\frac{\log\pi}{\log n}+\frac{2}{\pi\log n}\sum_{k=1}^{m-1}\frac{1}{k}+\frac{2}{(m\pi+\pi)\log n}$$

Taking the limit as $n \to \infty$, the first and third terms on the RHS of (3) tend to 0. We also have the following asymptotic behavior for the harmonic sums, where γ is the Euler-Mascheroni constant,

$$\sum_{k=2}^{m} \frac{1}{k} \sim \gamma + \log m \le \gamma + \log \frac{n}{\pi} \sim \log n, \sum_{k=1}^{m-1} \frac{1}{k} \sim \gamma + \log(m-1) \le \gamma + \log\left(\frac{n}{\pi} - 1\right) \sim \log n$$

and it follows from the squeeze theorem applied to (3) that

$$\lim_{n\to\infty} \frac{1}{\log n} \int_1^n \frac{|\sin x|}{x} dx = \frac{2}{\pi}$$