RMM - Geometry Marathon 1501 - 1600

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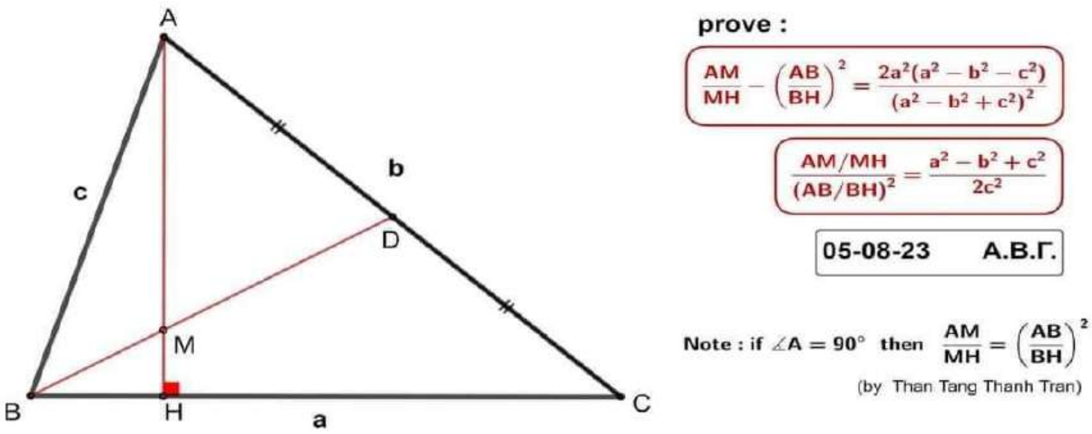
1501.



Prove that :

$$\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2=\frac{2a^2(a^2-b^2-c^2)}{(a^2-b^2+c^2)^2}\ \textit{and}\ \frac{AM/MH}{(AB/BH)^2}=\frac{a^2-b^2+c^2}{2c^2}$$

Proposed by Thanasis Gakopoulos-Greece

Solution by Soumava Chakraborty-Kolkata-India

Via Menelaus' theorem on Δ AHC with DMB as transversal,

$$\frac{BC}{BH}\cdot\frac{MH}{AM}\cdot\frac{AD}{CD}=1\Rightarrow\frac{AM}{MH}=\frac{a}{c\cos B}\Rightarrow\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2=\frac{a}{c\cos B}-\frac{c^2}{c^2\cos^2 B}$$

$$\Rightarrow\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2\overset{(*)}{=}\frac{a\cos B-c}{c\cos^2 B}$$

$$\text{Again, }\frac{2a^2(a^2-b^2-c^2)}{(a^2-b^2+c^2)^2}=\frac{2a^2(a^2-b^2-c^2)}{(2ac)^2\cos^2 B}=\frac{a^2-b^2+c^2-2c^2}{2c^2\cos^2 B}$$

$$=\frac{2ac\cos B-2c^2}{2c^2\cos^2 B}=\frac{a\cos B-c}{c\cos^2 B}\overset{\text{via }(*)}{=}\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2$$

$$\therefore\frac{AM}{MH}-\left(\frac{AB}{BH}\right)^2=\frac{2a^2(a^2-b^2-c^2)}{(a^2-b^2+c^2)^2}$$

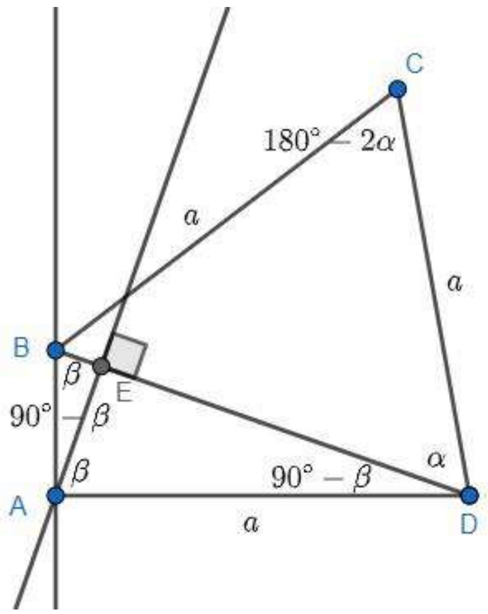
$$\text{Also, }\frac{AM/MH}{(AB/BH)^2}=\frac{\left(\frac{a}{c\cos B}\right)}{\left(\frac{c^2}{c^2\cos^2 B}\right)}=\frac{a\cos B}{c}=\frac{a(a^2-b^2+c^2)}{2ac.c}$$

$$\Rightarrow\frac{AM/MH}{(AB/BH)^2}=\frac{a^2-b^2+c^2}{2c^2}\ \text{(QED)}$$

1502. Let ABCD be the convex quadrilateral where the angle $\widehat{BAD}=\frac{\pi}{2},\hat{C}\in\left(0,\frac{\pi}{2}\right)$ with $AD=CD=BC=a>0$. $AE\perp BD, E\in(BD)$ and p is the semiperimeter of ABCD. Prove that:
$$\left(\sqrt{AC.\cos\hat{C}}+\sqrt{AE.\left|\sin(\hat{B}+\hat{D})\right|}\right)^2<a\sqrt{2}+\sqrt{\frac{a(2p-3a)}{2}}$$

Proposed by Radu Diaconu-Romania

Solution by Soumava Chakraborty-Kolkata-India



$$BD^2=a^2+a^2-2a^2\cos(180°-2\alpha)=4a^2\cos^2\alpha\Rightarrow BD\overset{(1)}{=}2a\cos\alpha$$
$(\because 180°-2\alpha>0\Rightarrow 0<\alpha<90°\Rightarrow\cos\alpha>0)$ and $AB^2=BD^2-AD^2$
$$=4a^2\cos^2\alpha-a^2\Rightarrow AB\overset{(2)}{=}a.\sqrt{4\cos^2\alpha-1}\text{ and }\therefore\Delta ABD\sim\Delta EAD\therefore\frac{BD}{AD}=\frac{AB}{AE}$$
$$\overset{\text{via (1) and (2)}}{\Rightarrow}\frac{2a\cos\alpha}{a}=\frac{a.\sqrt{4\cos^2\alpha-1}}{AE}\Rightarrow AE\overset{(*)}{=}\frac{a}{2}.\frac{\sqrt{4\cos^2\alpha-1}}{\cos\alpha}$$
Now, via Ptolemy's inequality, $AC.2a\cos\alpha<a.a.\sqrt{4\cos^2\alpha-1}+a^2$
$$\Rightarrow AC\overset{(**)}{<}\frac{a}{2}.\frac{1+\sqrt{4\cos^2\alpha-1}}{\cos\alpha}$$
$$\left(\sqrt{AC.\cos\hat{C}}+\sqrt{AE.\left|\sin(\hat{B}+\hat{D})\right|}\right)^2\overset{CBS}{\le}(AC+AE)\left(\cos\hat{C}+\left|\sin(\hat{B}+\hat{D})\right|\right)$$
$$\overset{\text{via }(*)\text{ and }(**)}{<}\frac{a\left(1+2\sqrt{4\cos^2\alpha-1}\right)}{2\cos\alpha}.\left(-\cos 2\alpha+|\sin(90°+2\alpha)|\right)$$
$$=\frac{a\left(1+2\sqrt{4\cos^2\alpha-1}\right)(-2\cos 2\alpha)}{2\cos\alpha}$$
$$(\because 180°-2\alpha<90°\Rightarrow 2\alpha>90°\Rightarrow\cos 2\alpha<0)\overset{?}{<}a\sqrt{2}$$

$$\Leftrightarrow \boxed{\sqrt{2}\cos\alpha \underbrace{\overset{?}{>}}_{(\bullet)} \left(1+2\sqrt{4\cos^2\alpha-1}\right)(-\cos 2\alpha)}$$

Let $\sqrt{4\cos^2\alpha-1}=t$ and $\because \alpha>45^\circ \Rightarrow \cos\alpha<\frac{1}{\sqrt{2}} \Rightarrow \sqrt{4\cos^2\alpha-1}<1$

$$\Rightarrow 0<t<1 \text{ and } \cos^2\alpha=\frac{t+1}{4} \Rightarrow \cos\alpha=\frac{\sqrt{t+1}}{2} \therefore (\bullet) \Leftrightarrow$$

$$\sqrt{2}.\frac{\sqrt{t+1}}{2}>\left(1-\frac{t+1}{2}\right)(1+2t) \Leftrightarrow \sqrt{\frac{t+1}{2}}>\left(\frac{1-t}{2}\right)(1+2t)$$

$$\Leftrightarrow \frac{t+1}{2}>\frac{(1-t)^2(1+2t)^2}{4}\ (\because (1-t),t>0) \Leftrightarrow 4t^4-4t^3-3t^2-1<0$$

$$\Leftrightarrow 4t^3(t-1)-3t^2-1<0 \to \text{true} \because 0<t<1 \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \left(\sqrt{AC.\cos\hat{C}}+\sqrt{AE.\left|\sin\left(\hat{B}+\hat{D}\right)\right|}\right)^2<a\sqrt{2}<a\sqrt{2}+\sqrt{\frac{a(2p-3a)}{2}}\ (QED)$$

1503. In $\Delta ABC,\Delta A'B'C'$ the following relationship holds:

$$(a+a')(b+b')(c+c')\ge 32\sqrt{RR'FF'}+4\left(\sqrt{RF}-\sqrt{R'F'}\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

$$(a+a')(b+b')(c+c')=$$

$$=abc+a'b'c'+(a'bc+ab'c+abc'+a'b'c+ab'c'+a'bc')\ge$$

$$\overset{AM-GM}{\ge} abc+a'b'c'+6\sqrt[6]{a'bc\cdot ab'c\cdot abc'\cdot a'b'c\cdot ab'c'\cdot a'bc'}=$$

$$=abc+a'b'c'+6\sqrt{abc\cdot a'b'c'}=4RF+4R'F'+6\sqrt{4RF\cdot 4R'F'}=$$

$$=4\left(\sqrt{RF}-\sqrt{R'F'}\right)^2+8\sqrt{RR'FF'}+24\sqrt{RR'FF'}=32\sqrt{RR'FF'}+4\left(\sqrt{RF}-\sqrt{R'F'}\right)^2$$

Equality holds for $a=b=c=a'=b'=c'$.

1504. If H −orthocenter in acute $\Delta ABC, AD, BE, CF$ −altitudes,

$HD = x, HE = y, HF = z$ then:

$$\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\ge\frac{2}{R}\cdot(R^2-r^2)$$

Proposed by Ertan Yildirim-Turkiye

Solution by Daniel Sitaru-Romania

$$cosB=\frac{BD}{AB}\Rightarrow BD=c\cdot cosB$$

$$tan\left(\frac{\pi}{2}-C\right)=\frac{HD}{BD}\Rightarrow HD=BD\cdot cosC=c\cdot cosBcotC=$$

$$=2RsinCcosB\cdot\frac{cosC}{sinC}=2RcosBcosC$$

$$\begin{cases}x=2RcosBcosC\\y=2RcosCcosA\\z=2RcosAcosB\end{cases}\Rightarrow\sum_{cyc}\frac{xy}{z}=\sum_{cyc}\frac{2RcosBcosC\cdot 2RcosCcosA}{2RcosAcosB}=$$

$$=2R\cdot\sum_{cyc}cos^2C=2R\cdot\frac{6R^2+4Rr+r^2-s^2}{2R^2}=$$

$$=\frac{1}{R}(6R^2+4Rr+r^2-s^2)\overset{GERRETSEN}{\ge}$$

$$\ge\frac{1}{R}(6R^2+4Rr+r^2-4R^2-4Rr-3r^2)=\frac{2}{R}\cdot(R^2-r^2)$$

Equality holds for $a=b=c$.

1505. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2}{w_b^2(w_b^5+h_c^5)}+\frac{w_b^2}{h_c^2(h_c^5+m_a^5)}+\frac{h_c^2}{m_a^2(m_a^5+w_b^5)}\ge\frac{2^{10}r^6}{81R^6(81R^5-2560r^5)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{am_a}\sum_{cyc}a^2\ge 2\sqrt{3}\Leftrightarrow\frac{1}{a^2m_a^2}\ge\frac{12}{\left(\sum_{cyc}a^2\right)^2}\Leftrightarrow$$

$$\left(\sum_{cyc} a^2\right)^2 - 3a^2(2b^2+2c^2-a^2) \ge 0 \Leftrightarrow \left(\sum_{cyc} a^2\right)^2 - 3a^2\left(2\sum_{cyc} a^2 - 3a^2\right) \ge 0$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)^2 - 6a^2\sum_{cyc} a^2 + 9a^4 \ge 0 \Leftrightarrow \left(\sum_{cyc} a^2 - 3a^2\right)^2 \ge 0$$

$$\Leftrightarrow (b^2+c^2-2a^2)^2 \ge 0 \to \text{true} \Rightarrow m_a^2 \le \frac{\left(\sum_{cyc} a^2\right)^2}{12a^2} \text{ and analogs} \to (1)$$

$$\text{Again, } m_a^2 \overset{?}{\le} \frac{b^3+c^3+abc}{4a} \Leftrightarrow a(2b^2+2c^2-a^2) \overset{?}{\le} b^3+c^3+abc$$

$$\Leftrightarrow \sum_{cyc} a^3 + abc \overset{?}{\ge} 2a(b^2+c^2)$$

$$\Leftrightarrow \sum_{cyc}(y+z)^3 + \prod_{cyc}(y+z) \overset{?}{\ge} 2(y+z)\left((z+x)^2+(x+y)^2\right)$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \Rightarrow x+y+z = 3s-2s = s \\ \Rightarrow a = y+z, b = z+x, c = x+y; x,y,z > 0 \end{matrix}\right)$$

$$\Leftrightarrow x^3 + y^2z + yz^2 \overset{?}{\ge} 3xyz \to \text{true via A}-\text{G} \therefore m_a^2 \le \frac{b^3+c^3+abc}{4a}$$

$$\Rightarrow m_a^3 \overset{Panaitopol}{\le} \frac{b^3+c^3+abc}{4a}.\frac{Rs}{a} = \frac{Rs}{4.16R^2r^2s^2}.b^2c^2(b^3+c^3+abc)$$

$$\Rightarrow m_a^3 \le \frac{1}{64Rr^2s}\left(b^2c^2\left(\sum_{cyc} a^3 + abc\right) - a^3b^2c^2\right) \Rightarrow m_a^5 \le$$

$$\frac{1}{256Rr^2s}.\left(b^2c^2\left(\sum_{cyc} a^3 + abc\right) - a^3b^2c^2\right)\left(2\sum_{cyc} a^2 - 3a^2\right) \text{ and analogs} \Rightarrow$$

$$\sum_{cyc} m_a^5 \le \frac{1}{256Rr^2s}.\left(\begin{matrix}\left(2\sum_{cyc} a^2\right)\left(\left(\sum_{cyc} a^3 + abc\right)\left(\sum_{cyc} b^2c^2\right) - 16R^2r^2s^2(2s)\right) \\ -9a^2b^2c^2\left(\sum_{cyc} a^3 + abc\right) + 3a^2b^2c^2\sum_{cyc} a^3\end{matrix}\right)$$

$$\overset{Goldstone}{\le} \frac{1}{256Rr^2s}\left(\begin{matrix}\left(2\sum_{cyc} a^2\right)\left(\begin{matrix}(2s(s^2-6Rr-r^2)+4Rrs)(4R^2s^2) \\ -16R^2r^2s^2(2s)\end{matrix}\right) \\ -9(16R^2r^2s^2)(2s(s^2-4Rr-r^2)+4Rrs) \\ +3(16R^2r^2s^2).2s(s^2-6Rr-r^2)\end{matrix}\right) =$$

$$\frac{1}{256Rr^2s}\left(\begin{matrix}\left(2\sum_{cyc} a^2\right)(8R^2s^3)(s^2-4Rr-7r^2) - 9(16R^2r^2s^2)(2s(s^2-4Rr-r^2)+4Rrs) \\ +3(16R^2r^2s^2).2s(s^2-6Rr-r^2)\end{matrix}\right)$$

$$\overset{\text{Leibnitz}}{\leq}\frac{(18R^2)(8R^2s^3)(s^2-4Rr-7r^2)-6(16R^2r^2s^2)\left(2s(s^2-4Rr-r^2)\right)-9(64R^3r^3s^3)}{256Rr^2s}$$

$$=\frac{48R^2s^3\left((3R^2-4r^2)s^2-r(12R^3+21R^2r-12Rr^2-12r^3)\right)}{256Rr^2s}$$

$$\overset{\text{Gerretsen}}{\leq}\frac{48R^2s^3\left((3R^2-4r^2)(4R^2+4Rr+3r^2)-r(12R^3+21R^2r-12Rr^2-12r^3)\right)}{256Rr^2s}$$

$$\therefore\sum_{cyc}m_a^5\leq\frac{48R^2s^3}{256Rr^2s}.4R(3R^3-7Rr^2-r^3)\to(2)$$

$$\text{Now,}\frac{m_a^2}{w_b^2(w_b^5+h_c^5)}+\frac{w_b^2}{h_c^2(h_c^5+m_a^5)}+\frac{h_c^2}{m_a^2(m_a^5+w_b^5)}$$

$$\geq\frac{h_a^2}{m_b^2(m_b^5+m_c^5)}+\frac{h_b^2}{m_c^2(m_c^5+m_a^5)}+\frac{h_c^2}{m_a^2(m_a^5+m_b^5)}\overset{\text{via (1)}}{\geq}$$

$$\frac{48r^2s^2}{\left(\sum_{cyc}a^2\right)^2}\left(\frac{\frac{b^2}{a^2}}{m_b^5+m_c^5}+\frac{\frac{c^2}{b^2}}{m_c^5+m_a^5}+\frac{\frac{a^2}{c^2}}{m_a^5+m_b^5}\right)\overset{\text{Bergstrom and Leibnitz}}{\geq}\frac{24r^2s^2\left(\sum_{cyc}\frac{b}{a}\right)^2}{81R^4.\sum_{cyc}m_a^5}$$

$$\overset{\text{via (2) and A-G}}{\geq}\frac{9.24r^2s^2.256Rr^2s}{81R^4.48R^2s^3.4R(3R^3-7Rr^2-r^3)}\overset{?}{\geq}\frac{1024r^6}{81R^6(81R^5-2560r^5)}$$

$$\Leftrightarrow 729t^5-96t^3+224t-23008\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(729t^4+1458t^3+2820t^2+5640t+11504)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\therefore\frac{m_a^2}{w_b^2(w_b^5+h_c^5)}+\frac{w_b^2}{h_c^2(h_c^5+m_a^5)}+\frac{h_c^2}{m_a^2(m_a^5+w_b^5)}\geq\frac{2^{10}r^6}{81R^6(81R^5-2560r^5)}$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)

1506. In any $\Delta\,ABC$, the following relationship holds:

$$\frac{m_a^5}{w_b^5(w_b+h_c)^2}+\frac{w_b^5}{h_c^5(h_c+m_a)^2}+\frac{h_c^5}{m_a^5(m_a+w_b)^2}\geq\frac{1}{3R^2}\left(\frac{2r}{R}\right)^{15}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^5}{w_b^5(w_b+h_c)^2}+\frac{w_b^5}{h_c^5(h_c+m_a)^2}+\frac{h_c^5}{m_a^5(m_a+w_b)^2}$$

$$\geq\frac{h_a^5}{m_b^5(m_b+m_c)^2}+\frac{h_b^5}{m_c^5(m_c+m_a)^2}+\frac{h_c^5}{m_a^5(m_a+m_b)^2}$$

$$\overset{Panaitopol}{\geq}\frac{\left(\frac{2rs}{a}\right)^5}{\left(\frac{Rs}{b}\right)^5(m_b+m_c)^2}+\frac{\left(\frac{2rs}{b}\right)^5}{\left(\frac{Rs}{c}\right)^5(m_c+m_a)^2}+\frac{\left(\frac{2rs}{c}\right)^5}{\left(\frac{Rs}{a}\right)^5(m_a+m_b)^2}$$

$$=\left(\frac{2r}{R}\right)^5\cdot\left(\frac{\left(\frac{b}{a}\right)^5}{(m_b+m_c)^2}+\frac{\left(\frac{c}{b}\right)^5}{(m_c+m_a)^2}+\frac{\left(\frac{a}{c}\right)^5}{(m_a+m_b)^2}\right)\overset{Holder}{\geq}$$

$$\left(\frac{2r}{R}\right)^5\cdot\frac{\left(\sum_{cyc}\frac{b}{a}\right)^5}{27\sum_{cyc}(m_b+m_c)^2}\overset{A-G}{\geq}\left(\frac{2r}{R}\right)^5\cdot\frac{243}{27.2\sum_{cyc}(m_b^2+m_c^2)}=\left(\frac{2r}{R}\right)^5\cdot\frac{9}{4\sum_{cyc}m_a^2}$$

$$=\left(\frac{2r}{R}\right)^5\cdot\frac{9}{4.\frac{3}{4}\sum_{cyc}a^2}\overset{Leibnitz}{\geq}\left(\frac{2r}{R}\right)^5\cdot\frac{9}{3.9R^2}=\frac{1}{3R^2}\frac{\left(\frac{2r}{R}\right)^{15}}{\left(\frac{2r}{R}\right)^{10}}\overset{Euler}{\geq}\frac{1}{3R^2}\frac{\left(\frac{2r}{R}\right)^{15}}{(1)^{10}}$$

$$\therefore\frac{m_a^5}{w_b^5(w_b+h_c)^2}+\frac{w_b^5}{h_c^5(h_c+m_a)^2}+\frac{h_c^5}{m_a^5(m_a+w_b)^2}\geq\frac{1}{3R^2}\left(\frac{2r}{R}\right)^{15}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1507. In any $\Delta\,ABC$ with $n\in\mathbb{N}^*$, the following relationship holds :

$$1+\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}}\geq 2\left(\frac{4R}{r}-3\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}}=\sum_{cyc}\frac{\cot^{4n}\frac{A}{2}}{\cot^{2n-1}\frac{A}{2}.\cot^{2n-1}\frac{B}{2}}=\sum_{cyc}\frac{\left(\cot^2\frac{A}{2}\right)^{2n}}{\left(\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}}\overset{Radon}{\geq}$$

$$\frac{\left(\sum_{cyc}\cot^2\frac{A}{2}\right)^{2n-1}.\left(\sum_{cyc}\cot^2\frac{A}{2}\right)}{\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}}\geq\frac{\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}.\left(\sum_{cyc}\cot^2\frac{A}{2}\right)}{\left(\sum_{cyc}\cot\frac{A}{2}\cot\frac{B}{2}\right)^{2n-1}}=\sum_{cyc}\cot^2\frac{A}{2}$$

$$=s^2\sum_{cyc}\frac{1}{r_a^2}=\frac{s^2\left(s^4-2rs^2(4R+r)\right)}{r^2s^4}\Rightarrow 1+\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}}\geq\frac{s^2-8Rr-r^2}{r^2}$$

$$\overset{?}{\geq}2\left(\frac{4R}{r}-3\right)\Leftrightarrow s^2-8Rr-r^2\overset{?}{\geq}8Rr-6r^2\Leftrightarrow s^2\overset{?}{\geq}16Rr-5r^2$$

$$\rightarrow \text{true via Gerretsen} \therefore 1+\sum_{cyc}\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}} \ge 2\left(\frac{4R}{r}-3\right)\ \forall\,\Delta\,ABC \text{ and}$$

$$\forall\, n \in \mathbb{N}^*, "=" \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1508. In any Δ ABC with $n \in \mathbb{N}^*$, the following relationship holds :

$$\sum_{cyc}\frac{\tan^{2n+1}\frac{A}{2}}{\tan^{2n-1}\frac{B}{2}} \ge 2\left(1-\frac{r}{R}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan^{2n+1}\frac{A}{2}}{\tan^{2n-1}\frac{B}{2}} = \sum_{cyc}\frac{\tan^{4n}\frac{A}{2}}{\tan^{2n-1}\frac{A}{2}.\tan^{2n-1}\frac{B}{2}} = \sum_{cyc}\frac{\left(\tan^2\frac{A}{2}\right)^{2n}}{\left(\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}} \overset{Radon}{\ge}$$

$$\frac{\left(\sum_{cyc}\tan^2\frac{A}{2}\right)^{2n-1}.\left(\sum_{cyc}\tan^2\frac{A}{2}\right)}{\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}} \ge \frac{\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}.\left(\sum_{cyc}\tan^2\frac{A}{2}\right)}{\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^{2n-1}}$$

$$= \sum_{cyc}\tan^2\frac{A}{2} = \frac{1}{s^2}\left((4R+r)^2-2s^2\right) = \frac{(4R+r)^2}{s^2} - 2 \overset{?}{\ge} 2\left(1-\frac{r}{R}\right)$$

$$\Leftrightarrow R(4R+r)^2 \underset{(*)}{\overset{?}{\ge}} (4R-2r)s^2$$

$$\text{Now, RHS of } (*) \overset{Rouche}{\le} (4R-2r)\left(2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\le} R(4R+r)^2$$

$$\Leftrightarrow R(4R+r)^2-(2R^2+10Rr-r^2)(4R-2r) \overset{?}{\ge} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2) \underset{(**)}{\overset{?}{\ge}} 2(4R-2r)(R-2r)\sqrt{R^2-2Rr}$$

$$\because R-2r \overset{Euler}{\ge} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$8R^2-12Rr+r^2 > 2(4R-2r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow \left(8R^2-12Rr+r^2\right)^2 - 4\left(R^2-2Rr\right)(4R-2r)^2 > 0$$

$$\Leftrightarrow r^2(4R+r)^2 > 0 \rightarrow \textit{true} \Rightarrow (**) \Rightarrow (*) \textit{ is true} \therefore \sum_{cyc}\frac{\tan^{2n+1}\frac{A}{2}}{\tan^{2n-1}\frac{B}{2}} \ge 2\left(1-\frac{r}{R}\right)$$

$$\forall\,\Delta\,ABC \text{ and } \forall\, n \in \mathbb{N}^*, "=" \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1509. In any Δ ABC, the following relationship holds :

$$\frac{\left(h_a^3+2h_aw_b(h_a+w_b)+w_b^3\right)^3}{h_a^4+2h_aw_b(h_a^2+w_b^2)+w_b^4}+\frac{\left(w_b^3+2w_bm_c(w_b+m_c)+m_c^3\right)^3}{w_b^4+2w_bm_c(w_b^2+m_c^2)+m_c^4}$$
$$+\frac{(m_c^3+2m_ch_a(m_c+h_a)+h_a^3)^3}{m_c^4+2m_ch_a(m_c^2+h_a^2)+h_a^4}\geq\frac{9.6^6.r^9}{R(9R^3-64r^3)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^4\overset{?}{\leq}54R^3(R-r)$$
$$\Leftrightarrow 2\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-16r^2s^2\overset{?}{\leq}54R^3(R-r)$$
$$\Leftrightarrow s^4-(8Rr+6r^2)s^2\underset{(1)}{\overset{?}{\leq}}27R^4-27R^3r-16R^2r^2-8Rr^3-r^4$$
$$\text{Now, LHS of (1)}\overset{\text{Gerretsen}}{\leq}(4R^2-4Rr-3r^2)s^2\overset{\text{Gerretsen}}{\leq}$$
$$(4R^2-4Rr-3r^2)(4R^2+4Rr+3r^2)\overset{?}{\leq}27R^4-27R^3r-16R^2r^2-8Rr^3-r^4$$
$$\Leftrightarrow 11t^4-27t^3+16t+8\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left((t-2)(11t^2+17t+24)+44\right)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$
$$\Rightarrow(1)\text{ is true}\therefore\sum_{cyc}a^4\overset{(*)}{\leq}54R^3(R-r)$$

$$\frac{\left(h_a^3+2h_aw_b(h_a+w_b)+w_b^3\right)^3}{h_a^4+2h_aw_b\left(h_a^2+w_b^2\right)+w_b^4}+\frac{\left(w_b^3+2w_bm_c(w_b+m_c)+m_c^3\right)^3}{w_b^4+2w_bm_c\left(w_b^2+m_c^2\right)+m_c^4}$$
$$+\frac{\left(m_c^3+2m_ch_a(m_c+h_a)+h_a^3\right)^3}{m_c^4+2m_ch_a\left(m_c^2+h_a^2\right)+h_a^4}\geq\sum_{cyc}\frac{\left(h_a^3+2h_ah_b(h_a+h_b)+h_b^3\right)^3}{m_a^4+2m_am_b\left(m_a^2+m_b^2\right)+m_b^4}$$
$$\overset{\text{Holder}}{\geq}\frac{\left(\sum_{cyc}\left(3h_ah_b(h_a+h_b)\right)\right)^3}{3\sum_{cyc}\left(m_a^4+\left(m_a^2+m_b^2\right)^2+m_b^4\right)}=\frac{\left(3.\frac{2r^2s^2}{R}.\sum_{cyc}\left(\frac{h_a+h_b}{h_c}\right)\right)^3}{3\left(4\sum_{cyc}m_a^4+2\sum_{cyc}m_b^2m_c^2\right)}$$
$$\overset{\substack{\text{Gerretsen + Euler}\\ \text{and}\\ \text{A-G}}}{\geq}\frac{\left(3.\frac{r^2.27Rr}{R}.6\right)^3}{3.6\sum_{cyc}m_a^4}=\frac{3^3.3^9.2^3.3^3.r^9}{\frac{3^3.6}{16}.\sum_{cyc}a^4}\overset{\text{via }(*)}{\geq}\frac{3^{11}.2^6.r^9}{2.3^3.R^3(R-r)}$$
$$\overset{?}{\geq}\frac{9.6^6.r^9}{R(9R^3-64r^3)}=\frac{3^8.2^6.r^9}{R(9R^3-64r^3)}\Leftrightarrow 9R^3-64r^3\overset{?}{\geq}2R^2(R-r)$$
$$\Leftrightarrow 7t^3+2t^2-64\overset{?}{\geq}0\Leftrightarrow(t-2)(7t^2+16t+32)\overset{?}{\geq}0\rightarrow\text{true}$$

$$\therefore \frac{\left(h_a^3+2h_aw_b(h_a+w_b)+w_b^3\right)^3}{h_a^4+2h_aw_b\left(h_a^2+w_b^2\right)+w_b^4}+\frac{\left(w_b^3+2w_bm_c(w_b+m_c)+m_c^3\right)^3}{w_b^4+2w_bm_c\left(w_b^2+m_c^2\right)+m_c^4}$$

$$+\frac{\left(m_c^3+2m_ch_a(m_c+h_a)+h_a^3\right)^3}{m_c^4+2m_ch_a\left(m_c^2+h_a^2\right)+h_a^4}\ge\frac{9.6^6.r^9}{R(9R^3-64r^3)}$$

$\forall\,\Delta\,ABC,'' ='' $ iff $\Delta\,ABC$ is equilateral (QED)

1510. In any $\Delta\,ABC$, the following relationship holds :

$$\frac{\left(h_a^4+2h_aw_b\left(h_a^2+w_b^2\right)+w_b^4\right)^n}{r_a^5+2r_a^2r_b^2(r_a+r_b)+r_b^5}+\frac{\left(w_b^4+2w_bm_c\left(w_b^2+m_c^2\right)+m_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}$$

$$+\frac{\left(m_c^4+2m_ch_a\left(m_c^2+h_a^2\right)+h_a^4\right)^n}{r_c^5+2r_c^2r_a^2(r_c+r_a)+r_a^5}\ge\frac{2^{n+4}.3^{5(n-1)}.r^{4n}}{(9R^3-64r^3)(3R^2-8r^2)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\sum_{cyc}\left(r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5\right)=2\sum_{cyc}r_a^5+2\sum_{cyc}\left(r_b^2r_c^2\left(\sum_{cyc}r_a-r_a\right)\right)$$

$$=2\left(\left(\sum_{cyc}r_a\right)^5-5\left(\sum_{cyc}r_a^2+\sum_{cyc}r_ar_b\right)\prod_{cyc}(r_b+r_c)\right)$$

$$+2(4R+r)\left(\left(\sum_{cyc}r_ar_b\right)^2-2rs^2\left(\sum_{cyc}r_a\right)\right)-2rs^2\left(\sum_{cyc}r_ar_b\right)$$

$$\overset{\text{via (i) and analogs}}{=}2\left((4R+r)^5-5\left((4R+r)^2-2s^2+s^2\right).64R^2.\frac{s^2}{16R^2}\right)$$

$$+2(4R+r)\left(s^4-2rs^2(4R+r)\right)-2rs^4$$

$$=2\left((4R+r)^5+24Rs^4-\left(320R^3+192R^2r+36Rr^2+2r^3\right)\right)s^2$$

$$\overset{\text{Gerretsen}}{\le}2\begin{pmatrix}(4R+r)^5+24R\left(4R^2+4Rr+3r^2\right)s^2\\-\left(320R^3+192R^2r+36Rr^2+2r^3\right)s^2\end{pmatrix}$$

$$=2\left((4R+r)^5-\left(224R^3+96R^2r-36Rr^2+2r^3\right)s^2\right)\overset{\text{Gerretsen}}{\le}$$

$$2(4R+r)^5-2\left(224R^3+96R^2r-36Rr^2+2r^3\right)\left(16R-5r^2\right)$$

$$\therefore \boxed{\sum_{cyc}\left(r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5\right)\le 2\left(1024R^5-2304R^4r+224R^2r^2+1216Rr^4+11r^5\right)} \to (1)$$

$$\text{Again, }\sum_{cyc}\left(h_b^4+2h_bh_c\left(h_b^2+h_c^2\right)+h_c^4\right)\overset{A-G}{\ge}$$

$$2\sum_{cyc}h_bh_c\left(h_b^2+h_c^2+h_bh_c\right)\overset{A-G}{\ge}\frac{6h_a^2h_b^2h_c^2}{4r^2s^2}\cdot\sum_{cyc}a^2\ge\frac{6.4r^4s^4}{4R^2r^2s^2}\cdot\frac{4s^2}{3}$$

$$=\frac{8r^2s^4}{R^2}\overset{\text{Gerretsen + Euler}}{\ge}\frac{2r^2}{R^2}(27Rr)^2$$

$$\boxed{\sum_{cyc}\left(h_b^4+2h_bh_c\left(h_b^2+h_c^2\right)+h_c^4\right)\ge 2.729r^4}\to(2)$$

$$\therefore\frac{\left(h_a^4+2h_aw_b\left(h_a^2+w_b^2\right)+w_b^4\right)^n}{r_a^5+2r_a^2r_b^2(r_a+r_b)+r_b^5}+\frac{\left(w_b^4+2w_bm_c\left(w_b^2+m_c^2\right)+m_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}$$

$$+\frac{\left(m_c^4+2m_ch_a\left(m_c^2+h_a^2\right)+h_a^4\right)^n}{r_c^5+2r_c^2r_a^2(r_c+r_a)+r_a^5}\ge\sum_{cyc}\frac{\left(h_b^4+2h_bh_c\left(h_b^2+h_c^2\right)+h_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}$$

$$\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}\left(h_b^4+2h_bh_c\left(h_b^2+h_c^2\right)+h_c^4\right)\right)^n}{3^{n-2}.\left(\sum_{cyc}\left(r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5\right)\right)}\overset{\text{via (1) and (2)}}{\ge}$$

$$\frac{\left(2.729r^4\right)^n}{3^{n-2}.2.(1024R^5-2304R^4r+224R^2r^2+1216Rr^4+11r^5)}\overset{?}{\ge}\frac{2^{n+4}.3^{5(n-1)}.r^{4n}}{(9R^3-64r^3)(3R^2-8r^2)}$$

$$\Leftrightarrow\left(3^7\right)\left(9R^3-64r^3\right)\left(3R^2-8r^2\right)\overset{?}{\ge}$$

$$32\left(1024R^5-2304R^4r+224R^2r^2+1216Rr^4+11r^5\right)$$

$$\Leftrightarrow 26281t^5+73728t^4-164632t^3-458816t^2+6144t+1119392\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)\left(26281t^3+178852t^2+445652t+608384\right)+657072\right)\overset{?}{\ge}0$$

$$\to\text{true}\because t\overset{\text{Euler}}{\ge}2\therefore\frac{\left(h_a^4+2h_aw_b\left(h_a^2+w_b^2\right)+w_b^4\right)^n}{r_a^5+2r_a^2r_b^2(r_a+r_b)+r_b^5}+$$

$$\frac{\left(w_b^4+2w_bm_c\left(w_b^2+m_c^2\right)+m_c^4\right)^n}{r_b^5+2r_b^2r_c^2(r_b+r_c)+r_c^5}+\frac{\left(m_c^4+2m_ch_a\left(m_c^2+h_a^2\right)+h_a^4\right)^n}{r_c^5+2r_c^2r_a^2(r_c+r_a)+r_a^5}$$

$$\ge\frac{2^{n+4}.3^{5(n-1)}.r^{4n}}{(9R^3-64r^3)(3R^2-8r^2)}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1511. In any ΔABC, the following relationship holds :

$$\frac{4}{R}\le\sum_{cyc}\frac{h_b+h_c}{r_a^2}\le\frac{R}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_b+h_c}{r_a^2}=\frac{2rs\left(\frac{b+c}{bc}\right)}{\frac{r^2s^2}{(s-a)^2}}=\frac{2rs}{4Rrs.r^2s^2}.\sum_{cyc}\left((s-a)^2\left(\sum_{cyc}ab-bc\right)\right)$$

$$=\frac{1}{2Rr^2s^2}.\left((s^2+4Rr+r^2).\sum_{cyc}(s^2-2sa+a^2)-\sum_{cyc}(s^2bc-2sabc+a^2bc)\right)$$

$$=\frac{(s^2+4Rr+r^2)\left(3s^2-4s^2+2(s^2-4Rr-r^2)\right)-s^2(s^2+4Rr+r^2)+2s.12Rrs-4Rrs.2s}{2Rr^2s^2}$$

$$=\frac{2r\left((4R-r)s^2-r(4R+r)^2\right)}{2Rr^2s^2}\therefore\sum_{cyc}\frac{h_b+h_c}{r_a^2}\overset{(*)}{=}\frac{(4R-r)s^2-r(4R+r)^2}{Rrs^2}$$

$$\therefore\sum_{cyc}\frac{h_b+h_c}{r_a^2}\le\frac{R}{r^2}\overset{via\,(*)}{\Leftrightarrow}\frac{(4R-r)s^2-r(4R+r)^2}{Rrs^2}\le\frac{R}{r^2}$$

$$\Leftrightarrow(R^2-4Rr+r^2)s^2+r^2(4R+r)^2\overset{(1)}{\ge}0$$

$\boxed{Case\ 1}$ $R^2-4Rr+r^2\ge0$ **and then, LHS of** $(1)\ge r^2(4R+r)^2>0\Rightarrow(1)$ **is true (strict inequality)**

$\boxed{Case\ 2}$ $R^2-4Rr+r^2<0$ **and then, LHS of** $(1)=-\left(-(R^2-4Rr+r^2)\right)s^2$

$$+r^2(4R+r)^2\overset{Gerretsen}{\ge}-\left(-(R^2-4Rr+r^2)\right)(4R^2+4Rr+3r^2)+r^2(4R+r)^2\overset{?}{\ge}0$$

$$\Leftrightarrow4t^4-12t^3+7t^2+4\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left((t-2)(4t^2+4t+7)+12\right)\overset{?}{\ge}0$$

$\rightarrow$ **true** $\because t\overset{Euler}{\ge}2\Rightarrow(1)$ **is true** $\therefore$ **combining both cases,** (1) **is true** $\forall\,\Delta\,ABC$

$$\therefore\sum_{cyc}\frac{h_b+h_c}{r_a^2}\le\frac{R}{r^2}\forall\,\Delta\,ABC$$

$$\text{Again, }\frac{4}{R}\le\sum_{cyc}\frac{h_b+h_c}{r_a^2}\overset{via\,(*)}{\Leftrightarrow}\frac{(4R-r)s^2-r(4R+r)^2}{Rrs^2}\ge\frac{4}{R}$$

$$\Leftrightarrow(4R-5r)s^2\overset{(2)}{\ge}r(4R+r)^2$$

$$\text{Now, }(4R-5r)s^2\overset{Gerretsen}{\ge}(4R-5r)(16Rr-5r^2)\overset{?}{\ge}r(4R+r)^2$$

$$\Leftrightarrow48R^2-108Rr+24r^2\overset{?}{\ge}0\Leftrightarrow12(4R-r)(R-2r)\overset{?}{\ge}0\rightarrow\text{true}\because R\overset{Euler}{\ge}2r$$

$$\Rightarrow(2)\text{ is true}\therefore\frac{4}{R}\le\sum_{cyc}\frac{h_b+h_c}{r_a^2}\text{ and so, }\frac{4}{R}\le\sum_{cyc}\frac{h_b+h_c}{r_a^2}\le\frac{R}{r^2}$$

$\forall\,\Delta\,ABC$, " = " **iff** $\Delta\,ABC$ **is equilateral (QED)**

1512. In any Δ ABC, the following relationship holds :

$$\frac{2}{r} \le \sum_{cyc} \frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\sum_{cyc} \frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2} \overset{via\ (i)\ and\ analogs}{\Leftrightarrow} \sum_{cyc} \frac{4R\cos^2\frac{A}{2}.a^2}{4r^2s^2} \le \frac{R}{r^2} \Leftrightarrow \sum_{cyc} a^2\cos^2\frac{A}{2} \le s^2$$

$$\Leftrightarrow \sum_{cyc}\left(a^3.\frac{s(s-a)}{abc}\right) \le s^2 \Leftrightarrow s\sum_{cyc} a^3 + 16r^2s^2 - 2\sum_{cyc} a^2b^2 \le 4Rrs^2$$

$$\Leftrightarrow s^2(s^2 - 6Rr - 3r^2) + 8r^2s^2 - (s^2 + 4Rr + r^2)^2 + 16Rrs^2 \le 2Rrs^2$$

$$\Leftrightarrow (2R + 3r)s^2 - r(4R + r)^2 \le 2Rs^2 \Leftrightarrow 3s^2 \le (4R + r)^2 \to \text{true via Trucht}$$

$$\therefore \sum_{cyc} \frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2}$$

$$\text{Again, } \sum_{cyc} \frac{r_b + r_c}{h_a^2} \overset{via\ (i)\ and\ analogs}{=} \sum_{cyc} \frac{4R\left(\frac{1}{h_a}\right)^2}{\sec^2\frac{A}{2}} \overset{Bergstrom}{\ge} \frac{4R.\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\frac{s^2 + (4R + r)^2}{s^2}}$$

$$= \frac{4Rs^2}{r^2(s^2 + (4R + r)^2)} \overset{?}{\ge} \frac{2}{r} \Leftrightarrow (2R - r)s^2 \underset{(*)}{\overset{?}{\ge}} r(4R + r)^2$$

$$\text{Now, } (2R - r)s^2 \overset{Gerretsen}{\ge} (2R - r)(16Rr - 5r^2) \overset{?}{\ge} r(4R + r)^2$$

$$\Leftrightarrow 8R^2 - 17Rr + 2r^2 \overset{?}{\ge} 0 \Leftrightarrow (R - r)(R - 2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{r_b + r_c}{h_a^2} \ge \frac{2}{r} \therefore \frac{2}{r} \le \sum_{cyc} \frac{r_b + r_c}{h_a^2} \le \frac{R}{r^2}$$

$\forall \Delta$ ABC, "=" iff Δ ABC is equilateral (QED)

1513. In any Δ ABC, the following relationships hold :

$$1.\ \frac{h_a^n}{w_b m_c}+\frac{w_b^n}{m_c h_a}+\frac{m_c^n}{h_a w_b}\ge\frac{4.3^{n-1}.r^n}{R^2}\ and$$

$$2.\ \frac{h_a^n}{w_b+m_c}+\frac{w_b^n}{m_c+h_a}+\frac{m_c^n}{h_a+w_b}\ge\frac{(3r)^n}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a^n}{w_b m_c}+\frac{w_b^n}{m_c h_a}+\frac{m_c^n}{h_a w_b}\ge\frac{m_a h_a^n}{m_a m_b m_c}+\frac{m_b h_b^n}{m_b m_c m_a}+\frac{m_c h_c^n}{m_c m_a m_b}\overset{m_a m_b m_c\le\frac{Rs^2}{2}}{\ge}$$

$$\frac{2}{Rs^2}\cdot\sum_{cyc}m_a h_a^n\overset{\text{Chebyshev}}{\ge}\frac{2}{3Rs^2}\cdot\left(\sum_{cyc}m_a\right)\left(\sum_{cyc}h_a^n\right)$$

$$(\because \text{WLOG assuming } a\ge b\ge c\Rightarrow m_a\le m_b\le m_c\ and\ h_a^n\le h_b^n\le h_c^n)$$

$$\overset{\substack{\text{Tereshin}\\+\\\text{Holder}}}{\ge}\frac{2}{3Rs^2}\cdot\sum_{cyc}\frac{b^2+c^2}{4R}\cdot\frac{1}{3^{n-1}}\cdot\left(\sum_{cyc}h_a\right)^n=\frac{2}{3Rs^2}\cdot\frac{\sum_{cyc}a^2}{2R}\cdot\frac{1}{3^{n-1}}\cdot\left(2rs\sum_{cyc}\frac{1}{a}\right)^n$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\sum_{cyc}a^2}{R^2s^2}\cdot\frac{1}{3^n}\cdot\left(2rs.\frac{9}{2s}\right)^n=\frac{\sum_{cyc}a^2}{R^2s^2}\cdot\frac{1}{3^n}.3^{2n}.r^n\overset{?}{\ge}\frac{4.3^{n-1}.r^n}{R^2}\Leftrightarrow 3\sum_{cyc}a^2\overset{?}{\ge}4s^2$$

$$\Leftrightarrow 3\sum_{cyc}a^2\overset{?}{\ge}\left(\sum_{cyc}a\right)^2\to\text{true}$$

$$\therefore\frac{h_a^n}{w_b m_c}+\frac{w_b^n}{m_c h_a}+\frac{m_c^n}{h_a w_b}\ge\frac{4.3^{n-1}.r^n}{R^2}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral}$$

$$\text{Again,}\frac{h_a^n}{w_b+m_c}+\frac{w_b^n}{m_c+h_a}+\frac{m_c^n}{h_a+w_b}\ge\frac{h_a^n}{m_b+m_c}+\frac{w_b^n}{m_c+m_a}+\frac{m_c^n}{m_a+m_b}$$

$$\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}h_a^n\right)\left(\sum_{cyc}\frac{1}{m_b+m_c}\right)$$

$$\left(\because\text{WLOG assuming } a\ge b\ge c\Rightarrow\frac{1}{m_b+m_c}\le\frac{1}{m_c+m_a}\le\frac{1}{m_a+m_b}\ and\ h_a^n\le h_b^n\le h_c^n\right)$$

$$\overset{\substack{\text{Bergstrom}\\+\\\text{Holder}}}{\ge}\frac{1}{3.3^{n-1}}\cdot\left(\sum_{cyc}h_a\right)^n\cdot\frac{9}{2\sum_{cyc}m_a}\overset{\text{Leuenbrger}}{\ge}\frac{1}{3^n}\cdot\left(2rs\sum_{cyc}\frac{1}{a}\right)^n\cdot\frac{9}{2(4R+r)}\overset{\substack{\text{Bergstrom}\\+\\\text{Euler}}}{\ge}$$

$$\frac{1}{3^n}\cdot\left(2rs\frac{9}{2s}\right)^n\cdot\frac{9}{2\left(\frac{9R}{2}\right)}=\frac{(3r)^n}{R}\ \therefore\frac{h_a^n}{w_b+m_c}+\frac{w_b^n}{m_c+h_a}+\frac{m_c^n}{h_a+w_b}\ge\frac{(3r)^n}{R}$$

$$\forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Proof of $m_a m_b m_c \leq \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{array}{c} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{array}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\ -576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\le\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\le}0$$

Now, LHS of $(\bullet)\overset{\text{Gerretsen}}{\le}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$

$-r^3(4R+r)^3\overset{?}{\le}0$

$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3\underset{(\bullet\bullet)}{\overset{?}{\ge}}20rs^4$

Now, LHS of $(\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)}s^2(16Rr-5r^2)(8R-16r)$

$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$ *and*

RHS of $(\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\le}}_{(b)}20rs^2(4R^2+4Rr+3r^2)$

$(a),(b)\Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$

$\ge 20rs^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\ge 0$

$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\ge}27r^2s^2$

Now, LHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\ge}}_{(c)}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\le}}_{(d)}27r^2(4R^2+4Rr+3r^2)$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\ \ge 27r^2(4R^2+4Rr+3r^2)$

$\Leftrightarrow 224t^3-587t^2+308t-60\ge 0\ \left(\text{where } t=\frac{R}{r}\right)$

$\Leftrightarrow (t-2)((t-2)(224t+309)+648)\ge 0\to$ true $\because t\overset{\text{Euler}}{\ge}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$

$\Rightarrow(\bullet)$ is true $\Rightarrow m_a^2m_b^2m_c^2\le\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\le\frac{Rs^2}{2}$ (QED)

1514. In any Δ ABC, the following relationship holds :

$$6\le\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\le 6\left(\frac{R}{2r}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a^2}{r_a^2}=4\sum_{cyc}\frac{s^2-2sa+a^2}{a^2}=\frac{4s^2\sum_{cyc}a^2b^2}{16R^2r^2s^2}-\frac{8s\sum_{cyc}ab}{4Rrs}+12$$

$$\Rightarrow\sum_{cyc}\frac{h_a^2}{r_a^2}=\frac{\sum_{cyc}a^2b^2-8Rr\sum_{cyc}ab+48R^2r^2}{4R^2r^2}\rightarrow(1)$$

$$\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\sum_{cyc}\frac{\sum_{cyc}h_a^2-h_a^2}{r_a^2}=\frac{\sum_{cyc}a^2b^2}{4R^2}\cdot\frac{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}{r^2s^4}-\sum_{cyc}\frac{h_a^2}{r_a^2}$$

$$\overset{\text{via (1)}}{=}\frac{\sum_{cyc}a^2b^2}{4R^2}\cdot\frac{s^4-2rs^2(4R+r)}{r^2s^4}-\frac{\sum_{cyc}a^2b^2-8Rr\sum_{cyc}ab+48R^2r^2}{4R^2r^2}$$

$$=\frac{-2r^2\sum_{cyc}a^2b^2-48R^2r^2s^2+8Rr\left(s^2\sum_{cyc}ab-\sum_{cyc}a^2b^2\right)}{4R^2r^2s^2}$$

$$\Rightarrow\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\frac{-s^4+\left(24R^2+4Rr-2r^2\right)s^2-r(4R+r)^3}{2R^2s^2}\rightarrow(i)$$

$$\therefore(i)\Rightarrow\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\ge 6\Leftrightarrow\frac{-s^4+\left(24R^2+4Rr-2r^2\right)s^2-r(4R+r)^3}{2R^2s^2}\ge 6$$

$$\Leftrightarrow -s^4+\left(12R^2+4Rr-2r^2\right)s^2-r(4R+r)^3\overset{(*)}{\ge}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}-\left(4R^2+4Rr+3r^2\right)s^2+\left(12R^2+4Rr-2r^2\right)s^2$$

$$-r(4R+r)^3\overset{?}{\ge}0\Leftrightarrow\left(8R^2-5r^2\right)s^2\underset{(**)}{\overset{?}{\ge}}r(4R+r)^3$$

$$\text{Again, }\left(8R^2-5r^2\right)s^2\overset{\text{Gerretsen}}{\ge}\left(8R^2-5r^2\right)\left(16Rr-5r^2\right)\overset{?}{\ge}r(4R+r)^3$$

$$\Leftrightarrow 16t^3-22t^2-23t+6\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left(16t^2+10t-3\right)\overset{?}{\ge}0\rightarrow\text{true}$$

$$\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\ge 6$$

$$\text{Also, }(i)\Rightarrow\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\le 6\left(\frac{R}{2r}\right)\Leftrightarrow\frac{-s^4+\left(24R^2+4Rr-2r^2\right)s^2-r(4R+r)^3}{2R^2s^2}$$

$$\le 6\left(\frac{R}{2r}\right)\Leftrightarrow rs^4+\left(6R^3-24R^2r-4Rr^2+2r^3\right)s^2+r^2(4R+r)^3\overset{(***)}{\ge}0$$

$$\text{Now, LHS of }(***)\overset{\text{Gerretsen}}{\ge}r\left(16Rr-5r^2\right)s^2+\left(6R^3-24R^2r-4Rr^2+2r^3\right)s^2$$

$$+r^2(4R+r)^3\overset{?}{\ge}0\Leftrightarrow\left(6R^3-24R^2r+12Rr^2-3r^3\right)s^2+r^2(4R+r)^3\underset{(****)}{\overset{?}{\ge}}0$$

$$\boxed{\text{Case 1}}\ 6R^3-24R^2r+12Rr^2-3r^3\ge 0\text{ and then : LHS of }(****)$$
$$\ge r^2(4R+r)^3>0\Rightarrow(****)\text{ is true (strict inequality)}$$

$$\boxed{\text{Case 2}}\ 6R^3-24R^2r+12Rr^2-3r^3<0\text{ and then : LHS of }(****)$$
$$=-\left(-\left(6R^3-24R^2r+12Rr^2-3r^3\right)s^2\right)+r^2(4R+r)^3$$

$$\overset{\text{Gerretsen}}{\ge}-\left(-\left(6R^3-24R^2r+12Rr^2-3r^3\right)\left(4R^2+4Rr+3r^2\right)\right)+r^2(4R+r)^3\overset{?}{\ge}0$$

$$\Leftrightarrow 12t^5 - 36t^4 + 17t^3 + 6t^2 + 18t - 4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(12t^3 + 12t^2 + 17t + 26) + 54\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****)$$

is true $\therefore$ combining both cases, $(****) \Rightarrow (***)$ is true $\forall \Delta ABC \therefore \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2}$

$$\leq 6\left(\frac{R}{2r}\right) \therefore 6 \leq \sum_{cyc} \frac{h_b^2 + h_c^2}{r_a^2} \leq 6\left(\frac{R}{2r}\right) \forall \Delta ABC,'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1515. In any Δ ABC, the following relationship holds :

$$6 \leq \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2} \leq 6\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{r_a^2}{h_a^2} = \frac{1}{4}\sum_{cyc} \frac{(a - s + s)^2}{(s-a)^2} = \frac{1}{4}\left(3 - 2s\sum_{cyc} \frac{1}{s-a} + s^2 \sum_{cyc} \frac{1}{(s-a)^2}\right)$$

$$= \frac{1}{4}\left(3 - \frac{2s(4Rr + r^2)}{r^2 s} + \frac{s^2}{r^4 s^2}\left(\left(\sum_{cyc}(s-b)(s-c)\right)^2 - 2(s-a)(s-b)(s-c)\sum_{cyc}(s-a)\right)\right)$$

$$= \frac{1}{4}\left(3 - \frac{2(4R+r)}{r} + \frac{r^2\left((4R+r)^2 - 2s^2\right)}{r^4}\right) \Rightarrow \sum_{cyc} \frac{r_a^2}{h_a^2} = \frac{8R^2 + r^2 - s^2}{2r^2} \to (1)$$

$$\therefore \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2} = \sum_{cyc} \frac{\sum_{cyc} r_a^2 - r_a^2}{h_a^2} = \frac{\left((4R+r)^2 - 2s^2\right)(s^2 - 4Rr - r^2)}{2r^2 s^2} - \sum_{cyc} \frac{r_a^2}{h_a^2}$$

$$\overset{\text{via (1)}}{=} \frac{\left((4R+r)^2 - 2s^2\right)(s^2 - 4Rr - r^2)}{2r^2 s^2} - \frac{8R^2 + r^2 - s^2}{2r^2}$$

$$\Rightarrow \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2} = \frac{-s^4 + (8R^2 + 16Rr + 2r^2)s^2 - r(4R+r)^3}{2r^2 s^2} \to (i)$$

$$\therefore (i) \Rightarrow \sum_{cyc} \frac{r_b^2 + r_c^2}{h_a^2} \geq 6 \Leftrightarrow \frac{-s^4 + (8R^2 + 16Rr + 2r^2)s^2 - r(4R+r)^3}{2r^2 s^2} \geq 6$$

$$\Leftrightarrow -s^4 + (8R^2 + 16Rr - 10r^2)s^2 - r(4R+r)^3 \overset{(*)}{\geq} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} -(4R^2 + 4Rr + 3r^2)s^2 + (8R^2 + 16Rr - 10r^2)s^2$$

$$-r(4R+r)^3 \overset{?}{\geq} 0 \Leftrightarrow (4R^2 + 12Rr - 13r^2)s^2 \underset{(**)}{\overset{?}{\geq}} r(4R+r)^3$$

$$\text{Again, } (4R^2+12Rr-13r^2)s^2 \overset{\text{Gerretsen}}{\geq} (4R^2+12Rr-13r^2)\binom{16Rr}{-5r^2} \overset{?}{\geq} r(4R+r)^3$$

$$\Leftrightarrow 4r(31R^2-70Rr+16r^2) \overset{?}{\geq} 0 \Leftrightarrow (31R-8r)(R-2r) \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{r_b^2+r_c^2}{h_a^2} \geq 6$$

$$\text{Also, (i)} \Rightarrow \sum_{cyc} \frac{r_b^2+r_c^2}{h_a^2} \leq 6\left(\frac{R}{2r}\right)^3 \Leftrightarrow \frac{-s^4+(8R^2+16Rr+2r^2)s^2-r(4R+r)^3}{2r^2s^2}$$

$$\leq 6\left(\frac{R}{2r}\right)^3 \Leftrightarrow 2rs^4+(3R^3-16R^2r-32Rr^2-4r^3)s^2+2r^2(4R+r)^3 \overset{(***)}{\geq} 0$$

$$\text{Now, LHS of } (***) \overset{\text{Gerretsen}}{\geq} 2r(16Rr-5r^2)s^2+(3R^3-16R^2r-32Rr^2-4r^3)s^2$$

$$+2r^2(4R+r)^3 \overset{?}{\geq} 0 \Leftrightarrow (3R^3-16R^2r-14r^3)s^2+2r^2(4R+r)^3 \underset{(****)}{\overset{?}{\geq}} 0$$

$\boxed{Case\ 1}$ $3R^3-16R^2r-14r^3 \geq 0$ and then : LHS of $(****)$
$\geq 2r^2(4R+r)^3 > 0 \Rightarrow (****)$ is true (strict inequality)
$\boxed{Case\ 2}$ $3R^3-16R^2r-14r^3 < 0$ and then : LHS of $(****)$

$$= -\left(-(3R^3-16R^2r-14r^3)s^2\right)+2r^2(4R+r)^3$$

$$\overset{\text{Gerretsen}}{\geq} -\left(-(3R^3-16R^2r-14r^3)(4R^2+4Rr+3r^2)\right)+2r^2(4R+r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow 12t^5-52t^4+73t^3-8t^2-32t-40 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(12t^3-4t^2+9t+44)+108\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****)$$

is true $\therefore$ combining both cases, $(****) \Rightarrow (***)$ is true $\forall\, \Delta\, ABC$

$$\therefore \sum_{cyc} \frac{r_b^2+r_c^2}{h_a^2} \leq 6\left(\frac{R}{2r}\right)^3$$

$$\therefore 6 \leq \sum_{cyc} \frac{r_b^2+r_c^2}{h_a^2} \leq 6\left(\frac{R}{2r}\right)^3 \quad \forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1516. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \frac{h_b^2+h_c^2}{r_a^2} \leq \sum_{cyc} \frac{r_b^2+r_c^2}{h_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{h_a^2}{r_a^2} = 4\sum_{cyc} \frac{s^2-2sa+a^2}{a^2} = \frac{4s^2\sum_{cyc} a^2b^2}{16R^2r^2s^2} - \frac{8s\sum_{cyc} ab}{4Rrs} + 12$$

$$\Rightarrow \sum_{cyc} \frac{h_a^2}{r_a^2} = \frac{\sum_{cyc} a^2b^2 - 8Rr\sum_{cyc} ab + 48R^2r^2}{4R^2r^2} \to (1)$$

$$\sum_{cyc}\frac{r_a^2}{h_a^2}=\frac{1}{4}\sum_{cyc}\frac{(a-s+s)^2}{(s-a)^2}=\frac{1}{4}\left(3-2s\sum_{cyc}\frac{1}{s-a}+s^2\sum_{cyc}\frac{1}{(s-a)^2}\right)$$

$$=\frac{1}{4}\left(3-\frac{2s(4Rr+r^2)}{r^2s}+\frac{s^2}{r^4s^2}\left(\begin{array}{c}\left(\sum_{cyc}(s-b)(s-c)\right)^2\\ -2(s-a)(s-b)(s-c)\sum_{cyc}(s-a)\end{array}\right)\right)$$

$$=\frac{1}{4}\left(3-\frac{2(4R+r)}{r}+\frac{r^2\left((4R+r)^2-2s^2\right)}{r^4}\right)\Rightarrow\sum_{cyc}\frac{r_a^2}{h_a^2}=\frac{8R^2+r^2-s^2}{2r^2}\to(2)$$

$$\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\sum_{cyc}\frac{\sum_{cyc}h_a^2-h_a^2}{r_a^2}=\frac{\sum_{cyc}a^2b^2}{4R^2}.\frac{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}{r^2s^4}-\sum_{cyc}\frac{h_a^2}{r_a^2}$$

$$\overset{via\ (1)}{=}\frac{\sum_{cyc}a^2b^2}{4R^2}.\frac{s^4-2rs^2(4R+r)}{r^2s^4}-\frac{\sum_{cyc}a^2b^2-8Rr\sum_{cyc}ab+48R^2r^2}{4R^2r^2}$$

$$=\frac{-2r^2\sum_{cyc}a^2b^2-48R^2r^2s^2+8Rr\left(s^2\sum_{cyc}ab-\sum_{cyc}a^2b^2\right)}{4R^2r^2s^2}$$

$$\Rightarrow\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\frac{-s^4+\left(24R^2+4Rr-2r^2\right)s^2-r(4R+r)^3}{2R^2s^2}\to(i)$$

$$\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}=\sum_{cyc}\frac{\sum_{cyc}r_a^2-r_a^2}{h_a^2}=\frac{\left((4R+r)^2-2s^2\right)\left(s^2-4Rr-r^2\right)}{2r^2s^2}-\sum_{cyc}\frac{r_a^2}{h_a^2}$$

$$\overset{via\ (2)}{=}=\frac{\left((4R+r)^2-2s^2\right)\left(s^2-4Rr-r^2\right)}{2r^2s^2}-\frac{8R^2+r^2-s^2}{2r^2}$$

$$\Rightarrow\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}=\frac{-s^4+\left(8R^2+16Rr+2r^2\right)s^2-r(4R+r)^3}{2r^2s^2}\to(ii)$$

$$\therefore(i),(ii)\Rightarrow\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\le\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}$$

$$\Leftrightarrow\frac{-s^4+\left(24R^2+4Rr-2r^2\right)s^2-r(4R+r)^3}{2R^2s^2}$$

$$\le\frac{-s^4+\left(8R^2+16Rr+2r^2\right)s^2-r(4R+r)^3}{2r^2s^2}$$

$$\Leftrightarrow\left(8R^4+16R^3r-22R^2r^2-4Rr^3+2r^4\right)s^2-\left(R^2-r^2\right)s^4$$

$$-r\left(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5\right)\overset{(*)}{\ge}0$$

$$\text{Now, LHS of }(*)\overset{Gerretsen}{\ge}\left(8R^4+16R^3r-22R^2r^2-4Rr^3+2r^4\right)s^2$$

$$-\left(4R^2+4Rr+3r^2\right)\left(R^2-r^2\right)s^2$$

$$-r\left(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5\right)$$

$$=\left(4R^4+12R^3r-21R^2r^2+5r^4\right)s^2$$

$$-r\left(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5\right)$$

$$\overset{Gerretsen}{\ge}\left(4R^4+12R^3r-21R^2r^2+5r^4\right)\left(16Rr-5r^2\right)$$

$$-r\left(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 31t^4-86t^3+38t^2+23t-6\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(31t^2+38t+66\right)+135\right)\overset{?}{\geq}0\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}$$

$$\therefore \sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\leq\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}\ \forall\ \Delta ABC,''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

1517. In any Δ ABC, the following relationship holds :

$$\frac{27r}{2Rp}\leq\sum_{cyc}\frac{\sin^3B+\sin^3C}{h_a}\leq\frac{27R}{8rp}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

For own convenience, $s\equiv p$

$$\sum_{cyc}\frac{\sin^3B+\sin^3C}{h_a}=\frac{1}{16R^3rs}\cdot\sum_{cyc}a\left(b^3+c^3\right)=\frac{1}{16R^3rs}\cdot\sum_{cyc}\left(ab\left(\sum_{cyc}a^2-a^2\right)\right)$$

$$=\frac{1}{16R^3rs}\cdot\left(\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right)-abc\left(\sum_{cyc}a\right)\right)=\frac{s^4-r^2(4R+r)^2-4Rrs^2}{8R^3rs}$$

$$\overset{\text{Gerretsen}}{\leq}\frac{s^2\left(4R^2+4Rr+3r^2\right)-r^2(4R+r)^2-4Rrs^2}{8R^3rs}\overset{\text{Gerretsen}}{\leq}$$

$$\frac{\left(4R^2+3r^2\right)\left(4R^2+4Rr+3r^2\right)-r^2(4R+r)^2}{8R^3rs}\overset{?}{\leq}\frac{27R}{8rs}$$

$$\Leftrightarrow 11t^4-16t^3-8t^2-4t-8\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left(11t^3+6t^2+4t+4\right)\overset{?}{\geq}0$$

$$\therefore\sum_{cyc}\frac{\sin^3B+\sin^3C}{h_a}\leq\frac{27R}{8rp}$$

$$\text{Again,}\ \sum_{cyc}\frac{\sin^3B+\sin^3C}{h_a}=\frac{2R}{8R^3}\cdot\sum_{cyc}\frac{b^3+c^3}{bc}\geq\frac{1}{4R^2}\sum_{cyc}\frac{bc(b+c)}{bc}=\frac{4s}{4R^2}=\frac{2s^2}{2R^2s}$$

$$\overset{\text{Gerretsen + Euler}}{\geq}\frac{27Rr}{2R^2s}=\frac{27r}{2Rs}\ \therefore\sum_{cyc}\frac{\sin^3B+\sin^3C}{h_a}\geq\frac{27r}{2Rp}$$

$$\therefore\frac{27r}{2Rp}\leq\sum_{cyc}\frac{\sin^3B+\sin^3C}{h_a}\leq\frac{27R}{8rp}\ \forall\ \Delta ABC,''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

1518. Let $n \geq 2$. Then, in Δ ABC, the following relationship holds :

$$2023\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}+\left(\frac{R}{r}\right)^n \geq 2^n+2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2 \Rightarrow an_a^2$$

$$=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a \therefore n_a^2=s^2-2r_ah_a$$

$$\Rightarrow a^2n_a^2 \overset{?}{\leq} 4(R-r)^2s^2a^2(s^2-2h_ar_a) \overset{?}{\leq} 4(R-r)^2s^2$$

$$\Leftrightarrow (4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right) \overset{?}{\leq} 4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2 \overset{?}{\geq} 0 \Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R\cos A-r)^2 \overset{?}{\geq} 0 \to \text{true} \therefore an_a \leq 2Rs-2rs \Rightarrow \frac{n_a}{h_a} \leq \frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$$

$$\Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r}-1 \text{ and analogs} \therefore 2023\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}+\frac{R^2-4r^2}{r^2}$$

$$\geq 2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{\left(\frac{R}{r}-1\right)(h_b+h_c)}}+\frac{R^2-4r^2}{r^2}$$

$$\geq 2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{\left(\frac{R}{r}-1\right)(m_b+m_c)}}+\frac{R^2-4r^2}{r^2} \overset{?}{\geq} 2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

$$\Leftrightarrow \boxed{\frac{R^2-4r^2}{r^2} \underset{(*)}{\overset{?}{\geq}} \frac{2023}{\sqrt[2023]{2}}\left(1-\frac{1}{\sqrt[2023]{\frac{R}{r}-1}}\right)\sum_{cyc}\sqrt[2023]{\frac{2m_a}{m_b+m_c}}}$$

$$\text{Now, }\sum_{cyc}\sqrt[2023]{\frac{2a}{b+c}}=\sum_{cyc}\sqrt[2023]{\frac{2a}{b+c}\cdot\underbrace{1.1\ldots1}_{2022\text{ terms}}} \overset{A-G}{\leq} \frac{1}{2023}\sum_{cyc}\left(\frac{2a}{b+c}+2+2020\right)$$

$$=\frac{1}{2023}\left(2\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{1}{b+c}\right)+6060\right)$$

$$=\frac{1}{2023}\left(2(2s)\frac{5s^2+4Rr+r^2}{2s(s^2+2Rr+r^2)}+6060\right)$$

$$=\frac{1}{2023}\left(\frac{10(s^2+2Rr+r^2)}{s^2+2Rr+r^2}-\frac{12Rr+8r^2}{s^2+2Rr+r^2}+6060\right)\overset{\text{Gerretsen}}{\leq}$$

$$\frac{1}{2023}\left(9+1-\frac{12Rr+8r^2}{4R^2+4Rr+3r^2+2Rr+r^2}+6060\right)$$

$$=\frac{1}{2023}\left(9+\frac{2R^2-3Rr-2r^2}{2R^2+3Rr+2r^2}+6060\right)\leq\frac{1}{2023}\left(9+\frac{R-2r}{r}+6060\right)$$

$$\left(\begin{array}{c}\frac{2R^2-3Rr-2r^2}{2R^2+3Rr+2r^2}\overset{?}{\leq}\frac{R-2r}{r}\Leftrightarrow 2\sigma^3-3\sigma^2-\sigma-2\overset{?}{\geq}0\ \left(\sigma=\frac{R}{r}\right)\\ \Leftrightarrow(\sigma-2)(2\sigma^2+\sigma+1)\overset{?}{\geq}0\to\text{true}\because\sigma\overset{\text{Euler}}{\geq}2\end{array}\right)$$

$\boxed{\sum_{cyc}\sqrt[2023]{\frac{2a}{b+c}}\leq\frac{1}{2023}\left(\frac{R}{r}+6067\right)}$ **and invoking it on a triangle with sides** $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$,

whose area via elementary calculations $=\frac{F}{3}$, **we arrive at :** $\sum_{cyc}\sqrt[2023]{\frac{2m_a}{m_b+m_c}}$

$$\leq\frac{1}{2023}\left(\frac{\frac{2m_a}{3}\cdot\frac{2m_b}{3}\cdot\frac{2m_c}{3}}{\frac{4F}{3}}\cdot\frac{\frac{\frac{2m_a}{3}+\frac{2m_b}{3}+\frac{2m_c}{3}}{2}}{\frac{F}{3}}+6067\right)$$

$$=\frac{1}{2023}\left(\frac{2m_am_bm_c(m_a+m_b+m_c)}{9r^2s^2}+6067\right)\overset{\substack{m_am_bm_c\leq\frac{Rs^2}{2}\\ \text{and}\\ m_a+m_b+m_c\leq 4R+r}}{\leq}$$

$$\frac{1}{2023}\left(\frac{Rs^2(4R+r)}{9r^2s^2}+6067\right)\Rightarrow\boxed{\sum_{cyc}\sqrt[2023]{\frac{2m_a}{m_b+m_c}}\leq\frac{1}{2023}\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)}$$

$$\Rightarrow\text{RHS of }(*)\leq\frac{1}{\sqrt[2023]{2}}\left(1-\frac{1}{t}\right)\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)\ \left(t=\sqrt[2023]{\frac{R}{r}-1}\overset{\text{Euler}}{\geq}1\right)$$

$$\leq\left(\frac{t-1}{t}\right)\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)\overset{?}{\leq}\frac{R^2-4r^2}{r^2}=\sigma^2-4$$

$$\Leftrightarrow(\sigma^2-4)t\overset{?}{\geq}(t-1)\left(\frac{4\sigma^2}{9}+\frac{\sigma}{9}+6067\right)$$

$$\Leftrightarrow t\left(\frac{5\sigma^2}{9}-\frac{\sigma}{9}-2-6069\right)+\frac{4\sigma^2}{9}+\frac{\sigma}{9}-2+6069\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{t(5\sigma+9)(\sigma-2)}{9}+\frac{(4\sigma+9)(\sigma-2)}{9}-6069(t-1)\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{t(5\sigma+9)+4\sigma+9}{9}.(t^{2023}-1)-6069(t-1)\overset{?}{\geq}0$$

$$\left(\because t^{2023}=\frac{R}{r}-1=\sigma-1\Rightarrow\sigma-2=t^{2023}-1\right)$$

$$\Leftrightarrow\boxed{(t-1)\left(\frac{t(5\sigma+9)+4\sigma+9}{9}.\left(t^{2022}+t^{2021}+\cdots+t+1\right)-6069\right)\underset{(**)}{\overset{?}{\geq}}0}$$

$$\text{Now},\because t\geq 1,\therefore\frac{t(5\sigma+9)+4\sigma+9}{9}.\left(t^{2022}+t^{2021}+\cdots+t+1\right)-6069$$

$$\geq\frac{5\sigma+9+4\sigma+9}{9}.\left(1^{2022}+1^{2021}+\cdots+1+1\right)-6069=2023(\sigma+2-3)$$

$$\Rightarrow \text{LHS of }(**)\geq 2023(t-1)(\sigma-1)\geq 0\because t\geq 1\text{ and }\sigma-1\overset{\text{Euler}}{\geq}1>0\Rightarrow(**)$$

$$\Rightarrow(*)\text{ is true}\therefore\frac{R^2-4r^2}{r^2}\geq 2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}-2023\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}\rightarrow(1)$$

$$\text{Let } f(n)=t^n-2^n\ \forall\, t=\frac{R}{r}\geq 2\ (t\rightarrow\text{fixed})\text{ and }\forall\, n\geq 2\text{ and then}:$$

$$f'(n)=t^n.(\ln t)-2^n.(\ln 2)\geq 0\because t^n\geq 2^n\text{ and }\ln t\geq\ln 2$$

$$\Rightarrow t^n.(\ln t)-2^n.(\ln 2)\geq 0\therefore f(n)\text{ is }\uparrow\forall\, n\geq 2\Rightarrow f(n)\geq f(2)$$

$$=\left(\frac{R}{r}\right)^2-4\Rightarrow\left(\frac{R}{r}\right)^n-2^n\geq\frac{R^2-4r^2}{r^2}\overset{\text{via }(1)}{\geq}$$

$$2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}-2023\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}$$

$$\therefore 2023\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}+\left(\frac{R}{r}\right)^n\geq 2^n+2023\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

$\forall\,\Delta ABC$ *and* $\forall\, n\geq 2$, " = " iff ΔABC is equi*lateral* (QED)

$$\boxed{\text{Proof of } m_a m_b m_c\leq\frac{Rs^2}{2}}$$

$$m_a^2m_b^2m_c^2=\frac{1}{64}\left(2b^2+2c^2-a^2\right)\left(2c^2+2a^2-b^2\right)\left(2a^2+2b^2-c^2\right)$$

$$\overset{(1)}{=}\frac{1}{64}\left\{-4\sum_{cyc}a^6+6\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)+3a^2b^2c^2\right\}$$

$$\text{Now},\sum_{cyc}a^6=\left(\sum_{cyc}a^2\right)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2m_b^2m_c^2$$

$$= \frac{1}{64}\begin{pmatrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{pmatrix}$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\begin{Bmatrix} -32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2 \\ -576Rrs^2(s^2-4Rr-r^2) - 432R^2r^2s^2 \end{Bmatrix}$$

$$= \frac{1}{16}\{s^6 - s^4(12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3\}$$

$$\leq \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2+12Rr-33r^2) - s^2(60R^2r^2+120Rr^3+33r^4) - r^3(4R+r)^3 \overset{(\bullet)}{\leq} 0$$

Now, LHS of $(\bullet) \overset{\text{Gerretsen}}{\leq} -s^4(8Rr-36r^2) - s^2(60R^2r^2+120Rr^3+33r^4)$
$-r^3(4R+r)^3 \overset{?}{\leq} 0$
$\Leftrightarrow s^4(8R-16r) + s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 20rs^4$

Now, LHS of $(\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)} s^2(16Rr-5r^2)(8R-16r)$
$+s^2(60R^2r+120Rr^2+33r^3) + r^2(4R+r)^3$ *and*
RHS of $(\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2(4R^2+4Rr+3r^2)$
$(a),(b) \Rightarrow$ in order to prove $(\bullet\bullet)$, it suffices to prove :

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$
$$\geq 20rs^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\geq 0$$
$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3\overset{(\bullet\bullet\bullet)}{\geq}27r^2s^2$$

Now, LHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet)\underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)}27r^2(4R^2+4Rr+3r^2)$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\geq 27r^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 224t^3-587t^2+308t-60\geq 0\ \left(\text{where } t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)((t-2)(224t+309)+648)\geq 0\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$
$$\Rightarrow(\bullet)\text{ is true}\Rightarrow m_a^2m_b^2m_c^2\leq\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\leq\frac{Rs^2}{2}\ (\text{QED})$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_a+m_b}}\leq\sqrt[2023]{\sum_{cyc}m_a(m_b+m_c).\sum_{cyc}\frac{1}{(m_a+m_b)(m_b+m_c)}.\left(\sum_{cyc}1\right)^{2021}}$$
$$=\sqrt[2023]{4.3^{2021}.\frac{(m_a+m_b+m_c)(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b)(m_b+m_c)(m_c+m_a)}}\overset{AM-GM}{\leq}$$
$$\leq\sqrt[2023]{4.3^{2021}.\frac{9}{8}}=\frac{3}{\sqrt[2023]{2}}.$$

Now, using AM − GM inequality and the known formula $n_a{}^2=s^2-2r_ah_a$, we have

$$n_a\leq\frac{n_a{}^2+r_a{}^2}{2r_a}=\frac{s^2+\left(s\tan\frac{A}{2}\right)^2}{2s\tan\frac{A}{2}}-h_a=\frac{s\sec^2\frac{A}{2}}{2\tan\frac{A}{2}}-h_a=\frac{s}{\sin A}-h_a=$$
$$=\frac{bc}{2r}-\frac{bc}{2R}=\frac{(R-r)bc}{2Rr}.$$

Using this result and the AM − GM inequality, we obtain

$$\sum_{cyc}\sqrt[2023]{\frac{n_a}{n_b+n_c}}\geq 3\sqrt[6069]{\frac{n_an_bn_c}{(n_a+n_b)(n_b+n_c)(n_c+n_a)}}$$
$$\geq 3\sqrt[6069]{\frac{(2Rr)^3.m_am_bm_c}{(R-r)^3abc(a+b)(b+c)(c+a)}}$$

$$\overset{m_a \ge \frac{(b+c)}{2}\cos\frac{A}{2}}{\ge} 3\sqrt[6069]{\frac{(Rr)^3.\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{(R-r)^3.abc}} = 3\sqrt[6069]{\frac{(Rr)^3.\frac{s}{4R}}{(R-r)^3.4Rsr}} = 3\sqrt[6069]{\frac{Rr^2}{16(R-r)^3}}$$

$$\ge 3\sqrt[6069]{\frac{r^3}{8(R-r)^3}} = 3\sqrt[2023]{\frac{r}{2(R-r)}} \overset{AM-GM}{\ge} \frac{3}{2023\sqrt[2023]{2}}\left(2024-\frac{R-r}{r}\right).$$

Using these results, it suffices to prove that

$$\frac{3}{\sqrt[2023]{2}}\left(2024-\frac{R-r}{r}\right)+\left(\frac{R}{r}\right)^n \ge 2^n+\frac{2023.3}{\sqrt[2023]{2}}$$

$$\Leftrightarrow \left(\frac{R}{r}-2\right)\left(\frac{R}{r}+2-\frac{3}{\sqrt[2023]{2}}\right)+\left(\frac{R}{r}\right)^2\left(\left(\frac{R}{r}\right)^{n-2}-1\right)-4\left(2^{n-2}-1\right)\ge 0.$$

which is true by Euler's inequality,

$R \ge 2r$. Equality holds iff ΔABC is equilateral.

1519. In any Δ ABC, the following relationships hold :

$$(a)\ \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a} \ge a^2+b^2+c^2+\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2} \text{ and}$$

$$(b)\ \sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}}+\frac{R^2}{r^2} \ge 4+\sum_{cyc}\sqrt{\frac{a}{b+c-a}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}-(a^2+b^2+c^2)-\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2}$$

$$=\frac{a^4}{ab}+\frac{b^4}{bc}+\frac{c^4}{ca}-(a^2+b^2+c^2)-\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2}$$

$$\overset{Bergstrom}{\ge} \frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}ab}-\sum_{cyc}a^2$$

$$-\frac{2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\cdot\left(\sum_{cyc}a^4+3\sum_{cyc}a^2b^2-2\sum_{cyc}\left(ab\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}\left(s^2-12Rr-3r^2\right)$$

$$-\frac{2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\cdot\begin{pmatrix}\left(\sum_{cyc}a^2\right)^2+\left(\sum_{cyc}ab\right)^2-2abc\sum_{cyc}a\\ -2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)+2abc\sum_{cyc}a\end{pmatrix}$$

$$=\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}\left(s^2-12Rr-3r^2\right)-\frac{2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\cdot\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)^2$$

$$=\frac{2\left(s^2-4Rr-r^2\right)}{\sum_{cyc}ab}\left(s^2-12Rr-3r^2\right)-\frac{2\left(s^2-12Rr-3r^2\right)^2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}$$

$$=2\left(s^2-12Rr-3r^2\right)\left(\frac{s^2-4Rr-r^2}{\sum_{cyc}ab}-\frac{s^2-12Rr-3r^2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\right)$$

$$\ge 2\left(s^2-12Rr-3r^2\right)\left(\frac{s^2-12Rr-3r^2}{\sum_{cyc}ab}-\frac{s^2-12Rr-3r^2}{\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2}\right)$$

$$\left(\because s^2-4Rr-r^2>s^2-12Rr-3r^2\overset{Euler}{\ge}16Rr-5r^2-12Rr-3r^2=4r(R-2r)\overset{Euler}{\ge}0\right)$$

$$=2\left(s^2-12Rr-3r^2\right)^2\left(\frac{\sum_{cyc}a^2-\sum_{cyc}ab+\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2+\left(\sum_{cyc}a\right)^2\right)}\right)\ge 0$$

$$\therefore\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\ge a^2+b^2+c^2+\frac{\sum_{cyc}(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2},\ ''=''\text{ iff }\Delta\,ABC\text{ is equilateral}$$

Now, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$
$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s\left(b^2+c^2\right)-2sbc=an_a^2+a\left(as-s^2\right)\Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore n_a^2=s^2-2r_ah_a$$

$$\Rightarrow a^2n_a^2\overset{?}{\le}4(R-r)^2s^2a^2\left(s^2-2h_ar_a\right)\overset{?}{\le}4(R-r)^2s^2$$

$$\Leftrightarrow\left(4R^2\sin^2A\right)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\le}4\left(R^2-2Rr+r^2\right)s^2$$

$$\Leftrightarrow R^2\left(1-\sin^2A\right)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\ge}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\ge}0\to\text{true}\therefore an_a\le 2Rs-2rs\Rightarrow\frac{n_a}{h_a}\le\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$$

$$\Rightarrow \frac{n_a}{h_a} \le \frac{R}{r} - 1 \text{ and analogs} \to (1)$$

$$\text{Triangle inequality} \Rightarrow g_a \le AI + r \overset{?}{\le} w_a \Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{2abc\cos\frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin\frac{A}{2}} + r \overset{?}{\le} \frac{8Rrs\cos\frac{A}{2}}{4R(b+c)\sin\frac{A}{2}\cos\frac{A}{2}} \Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} \frac{a+b+c}{(b+c)\sin\frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin\frac{A}{2}} + 1 \overset{?}{\le} + \frac{a}{(b+c)\sin\frac{A}{2}} + \frac{1}{\sin\frac{A}{2}} \Leftrightarrow (b+c)\sin\frac{A}{2} \overset{?}{\le} a$$

$$\Leftrightarrow 4R\cos\frac{A}{2}\cos\frac{B-C}{2}\sin\frac{A}{2} \overset{?}{\le} 4R\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow \cos\frac{B-C}{2} \overset{?}{\le} 1 \to \text{true}$$

$$\therefore g_a \le w_a \overset{A-G}{\le} \sqrt{s(s-a)} \le m_a \text{ and analogs}$$

$$\therefore \sum_{cyc}\sqrt{\frac{m_a}{n_a + g_a - m_a}} \ge \sum_{cyc}\sqrt{\frac{h_a}{n_a}} \overset{\text{via (1)}}{\ge} \sum_{cyc}\sqrt{\frac{h_a}{h_a\left(\frac{R}{r}-1\right)}}$$

$$\Rightarrow \sum_{cyc}\sqrt{\frac{m_a}{n_a + g_a - m_a}} \ge 3\sqrt{\frac{r}{R-r}} \to (2)$$

$$\text{Also,} \sum_{cyc}\sqrt{\frac{a}{b+c-a}} \overset{CBS}{\le} \sqrt{\sum_{cyc} a} \cdot \sqrt{\sum_{cyc}\frac{1}{2(s-a)}} = \sqrt{\frac{2s(4Rr+r^2)}{2r^2s}}$$

$$= \sqrt{\frac{4R+r}{r}} \overset{?}{\le} \frac{R^2-4r^2}{r^2} + 3\sqrt{\frac{r}{R-r}} \Leftrightarrow \frac{R^2-4r^2}{r^2} \overset{?}{\ge} \frac{\frac{4R+r}{r} - \frac{9r}{R-r}}{\sqrt{\frac{4R+r}{r}} + 3\sqrt{\frac{r}{R-r}}}$$

$$\Leftrightarrow \frac{(R-2r)(R+2r)}{r^2} \overset{?}{\ge} \frac{\frac{(4R+5r)(R-2r)}{r(R-r)}}{\sqrt{\frac{4R+r}{r}} + 3\sqrt{\frac{r}{R-r}}} \text{ and } \because R - 2r \overset{\text{Euler}}{\ge} 0 \therefore \text{it suffices to prove :}$$

$$\frac{(R+2r)(R-r)}{r}\left(\sqrt{\frac{4R+r}{r}} + 3\sqrt{\frac{r}{R-r}}\right) \overset{?}{>} 4R+5r$$

$$\Leftrightarrow \frac{(R+2r)^2(R-r)^2}{r^2}\left(\frac{4R+r}{r} + \frac{9r}{R-r} + 6\sqrt{\frac{4R+r}{R-r}}\right) \underbrace{\overset{?}{>}}_{(*)} 4R+5r$$

$$\text{Now,} \frac{4R+r}{R-r} \overset{?}{\ge} 9 \cdot \frac{4r^2}{R^2} \Leftrightarrow 4t^3 + t^2 - 36t + 36 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(4t^2 + 9(t-2)\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\therefore \text{LHS of } (*) > \frac{(R+2r)^2(R-r)^2}{r^2}\left(\frac{4R+r}{r} + \frac{9r}{R-r} + \frac{36r}{R}\right)$$

$$=\frac{(R+2r)^2(R-r)^2}{r^2}\left(\frac{R(4R+r)(R-r)+9Rr^2+36r^2(R-r)}{Rr(R-r)}\right)\overset{?}{>}4R+5r$$

$$\Leftrightarrow 4t^6+9t^5+35t^4+64t^3-56t^2-201t+144\overset{?}{>}0$$

$$\Leftrightarrow (t-2)(4t^5+17t^4+69t^3+202t^2+348t+495)+1134\overset{?}{>}0\to \text{true}$$

$$\Rightarrow (*)\text{ is true}\Rightarrow \sum_{cyc}\sqrt{\frac{a}{b+c-a}}\le\frac{R^2-4r^2}{r^2}+3\sqrt{\frac{r}{R-r}}\overset{\text{via }(2)}{\le}\sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}}$$

$$\therefore \sum_{cyc}\sqrt{\frac{m_a}{n_a+g_a-m_a}}+\frac{R^2}{r^2}\ge 4+\sum_{cyc}\sqrt{\frac{a}{b+c-a}},$$

$''=''$ iff Δ ABC is equilateral (QED)

1520.

In any Δ ABC and $\forall$ n $\in\mathbb{N}$: n $\ge$ 2, the following relationship hold :

$$\frac{h_a^n}{w_b^2m_c(w_b^2+m_c^2)}+\frac{w_b^n}{m_c^2h_a(m_c^2+h_a^2)}+\frac{m_c^n}{h_a^2w_b(h_a^2+w_b^2)}\ge\frac{16.3^{n-4}.r^n}{81R^5-2560r^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a^n}{w_b^2m_c(w_b^2+m_c^2)}+\frac{w_b^n}{m_c^2h_a(m_c^2+h_a^2)}+\frac{m_c^n}{h_a^2w_b(h_a^2+w_b^2)}$$

$$\ge\frac{h_a^n}{m_b^2m_c(m_b^2+m_c^2)}+\frac{h_b^n}{m_c^2m_a(m_c^2+m_a^2)}+\frac{h_c^n}{m_a^2m_b(m_a^2+m_b^2)}$$

$$\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}h_a\right)^n}{3^{n-2}.\sum_{cyc}\left(m_b^2m_c\left(\sum_{cyc}m_a^2-m_a^2\right)\right)}$$

$$=\frac{\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{3^{n-2}.\left(\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}m_b^2m_c\right)-m_am_bm_c\sum_{cyc}m_am_b\right)}$$

$$\overset{\substack{\text{Bergstrom}\\+\\\text{A-G}}}{\ge}\frac{\left(2rs.\frac{9}{2s}\right)^n}{3^{n-2}.\left(\frac{3}{4}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}m_a^3\right)-m_am_bm_c.\sum_{cyc}h_ah_b\right)}$$

$$\overset{\text{Leibnitz}}{\ge}\frac{3^{2n}.r^n}{3^{n-2}.\left(\frac{3}{4}.9R^2.\left(\left(\sum_{cyc}m_a\right)^3-3(m_a+m_b)(m_b+m_c)(m_c+m_a)\right)-m_am_bm_c.\sum_{cyc}\frac{bc.ca}{4R^2}\right)}$$

$$\overset{\substack{\text{Leuenberger}\\+\\\text{Cesaro}}}{\ge}\frac{3^{n+2}.r^n}{\frac{27R^2}{4}((4R+r)^3-24m_am_bm_c)-m_am_bm_c.\frac{4Rrs(2s)}{4R^2}}$$

$$\geq \frac{3^{n+2}.r^n}{\frac{27R^2}{4}((4R+r)^3-24h_ah_bh_c)-h_ah_bh_c.\frac{2rs^2}{R}}$$

$$=\frac{3^{n+2}.r^n}{\frac{27R^2}{4}\left((4R+r)^3-24.\frac{2r^2s^2}{R}\right)-\frac{2r^2s^2}{R}.\frac{2rs^2}{R}}$$

$$\overset{\text{Gerretsen + Euler}}{\geq}\frac{3^{n+2}.r^n}{\frac{27R^2}{4}\left((4R+r)^3-24.\frac{r^2.27Rr}{R}\right)-\frac{r^2.27Rr}{R}.\frac{r.27Rr}{R}}$$

$$=\frac{4.3^{n-1}.r^n}{R^2((4R+r)^3-24.27r^3)-108r^5}\overset{?}{\geq}\frac{16.3^{n-4}.r^n}{81R^5-2560r^5}$$

$$\Leftrightarrow 27(81R^5-2560r^5)\overset{?}{\geq}4\left(R^2((4R+r)^3-24.27r^3)-108r^5\right)$$

$$\Leftrightarrow 1931t^5-192t^4-48t^3+2588t^2-68688\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1931t^4+3670t^3+7292t^2+17172t+34344)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2$$

$$\therefore \frac{h_a^n}{w_b^2m_c(w_b^2+m_c^2)}+\frac{w_b^n}{m_c^2h_a(m_c^2+h_a^2)}+\frac{m_c^n}{h_a^2w_b(h_a^2+w_b^2)}\geq\frac{16.3^{n-4}.r^n}{81R^5-2560r^5}$$

$\forall\,\Delta\,ABC$ and $\forall\, n\in\mathbb{N}: n\geq 2$, " = " iff $\Delta\,ABC$ is equilateral (QED)

1521.

In any $\Delta\,ABC$ and $\forall\, n\in\mathbb{N}: n\geq 2$, the following relationship hold :

$$\frac{h_a^n}{r_b^2r_c(r_b^3+r_c^3)}+\frac{w_b^n}{r_c^2r_a(r_c^3+r_a^3)}+\frac{m_c^n}{r_a^2r_b(r_a^3+r_b^3)}\geq\frac{32.3^{n-5}.r^n}{3(9R^3-64r^3)^2-128r^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\sum_{cyc}r_b^2r_c(r_b^3+r_c^3)=\sum_{cyc}\left(r_b^2r_c\left(\sum_{cyc}r_a^3-r_a^3\right)\right)$$

$$=\left(\sum_{cyc}r_a^3\right)\left(\sum_{cyc}r_b^2r_c\right)-r_ar_br_c\left(\sum_{cyc}r_a^2r_b\right)\overset{A-G}{\leq}\left(\sum_{cyc}r_a^3\right)^2-3r_a^2r_b^2r_c^2$$

$$=\left((4R+r)^3-3\prod_{cyc}(r_b+r_c)\right)^2-3r^2s^4$$

$$\overset{via\ (i)\ and\ analogs}{=}\left((4R+r)^3-3.64R^3.\frac{s^2}{16R^2}\right)^2-3r^2s^4$$

$$=(4R+r)^6+\left(144R^2-3r^2\right)s^4-24R(4R+r)^3s^2\overset{\substack{Euler\\and\\Mitrinovic}}{\leq}$$

$$(4R+r)^4.\frac{81R^2}{4}+\left(\left(144R^2-3r^2\right).\frac{27R^2}{4}-24R(4R+r)^3\right)s^2$$

$$=(4R+r)^4.\frac{81R^2}{4}-\frac{3s^2}{4}\left(752R^4+1536R^3r+411R^2r^2+32Rr^3\right)\overset{Gerretsen+Euler}{\leq}$$

$$(4R+r)^4.\frac{81R^2}{4}-\frac{3.27Rr}{8}\left(752R^4+1536R^3r+411R^2r^2+32Rr^3\right)$$

$$\therefore\ \boxed{32\sum_{cyc}r_b^2r_c\left(r_b^3+r_c^3\right)\overset{(*)}{\leq}81\left(8R^2(4R+r)^4-4Rr\left(752R^4+1536R^3r+411R^2r^2+32Rr^3\right)\right)}$$

$$\text{Now,}\frac{h_a^n}{r_b^2r_c\left(r_b^3+r_c^3\right)}+\frac{w_b^n}{r_c^2r_a\left(r_c^3+r_a^3\right)}+\frac{m_c^n}{r_a^2r_b\left(r_a^3+r_b^3\right)}\geq$$

$$\frac{h_a^n}{r_b^2r_c\left(r_b^3+r_c^3\right)}+\frac{h_b^n}{r_c^2r_a\left(r_c^3+r_a^3\right)}+\frac{h_c^n}{r_a^2r_b\left(r_a^3+r_b^3\right)}\overset{Holder}{\geq}\frac{32\left(\sum_{cyc}h_a\right)^n}{3^{n-2}.32\sum_{cyc}r_b^2r_c\left(r_b^3+r_c^3\right)}$$

$$\overset{via\ (*)}{\geq}\frac{32\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{3^{n-2}.81\left(8R^2(4R+r)^4-4Rr(752R^4+1536R^3r+411R^2r^2+32Rr^3)\right)}\overset{Bergstrom}{\geq}$$

$$\frac{32\left(2rs.\frac{9}{2s}\right)^n}{3^{n-2}.81\left(8R^2(4R+r)^4-4Rr(752R^4+1536R^3r+411R^2r^2+32Rr^3)\right)}\overset{?}{\geq}$$

$$\frac{32.3^{n-5}.r^n}{3(9R^3-64r^3)^2-128r^6}$$

$$\Leftrightarrow\frac{2187}{81\left(8R^2(4R+r)^4-4Rr(752R^4+1536R^3r+411R^2r^2+32Rr^3)\right)}$$

$$\overset{?}{\geq}\frac{1}{3(9R^3-64r^3)^2-128r^6}\Leftrightarrow 81\left(9R^3-64r^3\right)^2-27.128r^6\overset{?}{\geq}$$

$$8R^2(4R+r)^4-4Rr\left(752R^4+1536R^3r+411R^2r^2+32Rr^3\right)$$

$$\Leftrightarrow 4513t^6+960t^5+5376t^4-91796t^3+120t^2+328320\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left((t-2)\left(4513t^4+19012t^3+63372t^2+85644t+89208\right)+14256\right)$$

$$\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{Euler}{\geq}2\ \therefore\frac{h_a^n}{r_b^2r_c\left(r_b^3+r_c^3\right)}+\frac{w_b^n}{r_c^2r_a\left(r_c^3+r_a^3\right)}+\frac{m_c^n}{r_a^2r_b\left(r_a^3+r_b^3\right)}$$

$$\geq\frac{32.3^{n-5}.r^n}{3(9R^3-64r^3)^2-128r^6}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$

1522. In any Δ ABC, the following relationship holds :

$$\frac{27r}{2Rp} \le \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} \le \frac{27R^2}{16r^2 p}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

For own convenience, $s \equiv p$

$$\sum_{cyc} a^4 = 2\sum_{cyc} a^2b^2 - 16r^2s^2 = 2(s^2+4Rr+r^2)^2 - 32Rrs^2 - 16r^2s^2$$

$$= 2s^4 - (16Rr+12r^2)s^2 + 2r^2(4R+r)^2 \overset{\text{Gerretsen}}{\le} (8R^2+8Rr+6r^2)s^2$$

$$-(16Rr+12r^2)s^2 + 2r^2(4R+r)^2 = (8R^2-8Rr-6r^2)s^2 + 2r^2(4R+r)^2$$

$$\overset{\text{Gerretsen}}{\le} (8R^2-8Rr-6r^2)(4R^2+4Rr+3r^2) + 2r^2(4R+r)^2 \overset{?}{\le} 54R^3(R-r)$$

$$\Leftrightarrow 11t^4 - 27t^3 + 16t + 8 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(11t^2+17t+24)+44\big) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow \sum_{cyc} a^4 \le 54R^3(R-r) \text{ and } \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} = \frac{1}{8R^3rs}\cdot\sum_{cyc}\big((b^3+c^3)(s-a)\big)$$

$$= \frac{1}{8R^3rs}\cdot\sum_{cyc}\left(\left(\sum_{cyc} a^3 - a^3\right)(s-a)\right)$$

$$= \frac{1}{8R^3rs}\cdot\left(\left(\sum_{cyc} a^3\right)\left(\sum_{cyc}(s-a)\right) - s\sum_{cyc} a^3 + \sum_{cyc} a^4\right) \le \frac{54R^3(R-r)}{8R^3rs} \overset{?}{\le} \frac{27R^2}{16r^2s}$$

$$\Leftrightarrow R^2 \overset{?}{\ge} 4r(R-r) \Leftrightarrow (R-2r)^2 \overset{?}{\ge} 0 \to \text{true} \therefore \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} \le \frac{27R^2}{16r^2p}$$

$$\text{Again}, \sum_{cyc} \frac{\sin^3 B + \sin^3 C}{r_a} = \frac{1}{8R^3rs}\cdot\sum_{cyc}\big((b^3+c^3)(s-a)\big)$$

$$= \frac{1}{8R^3rs}\cdot\sum_{cyc}\left(\left(\sum_{cyc} a^3 - a^3\right)(s-a)\right)$$

$$= \frac{1}{8R^3rs}\cdot\left(\left(\sum_{cyc} a^3\right)\left(\sum_{cyc}(s-a)\right) - s\sum_{cyc} a^3 + \sum_{cyc} a^4\right) = \frac{1}{8R^3rs}\cdot\sum_{cyc} a^4$$

$$\geq \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\sum_{cyc}a^2\right)^2 \geq \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\frac{1}{3}\left(\sum_{cyc}a\right)^2\right)^2 = \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\frac{4s^2}{3}\right)^2$$

$$\overset{\text{Gerretsen + Euler}}{\geq} \frac{1}{8R^3rs}\cdot\frac{1}{3}\left(\frac{2.27Rr}{3}\right)^2 = \frac{108R^2r^2}{8R^3rs} = \frac{27r}{2Rs} \therefore \sum_{cyc}\frac{\sin^3 B + \sin^3 C}{r_a} \geq \frac{27r}{2Rp}$$

$$\therefore \frac{27r}{2Rp} \leq \sum_{cyc}\frac{\sin^3 B + \sin^3 C}{r_a} \leq \frac{27R^2}{16r^2p}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC\ \text{is equilateral (QED)}$$

1523. In any Δ ABC, the following relationships hold :

$$\text{(i)} \sum_{cyc}a^4 + abc(a+b+c) \geq \sum_{cyc}ab(a^2+b^2) + \frac{r^4(R-2r)}{16R}\ \text{and}$$

$$\text{(ii)}\ \frac{a+b+c}{3} + \frac{R^3-4r^3}{2r^2} \geq \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}a^4 + abc(a+b+c) \geq \sum_{cyc}ab(a^2+b^2) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow 2\sum_{cyc}a^2b^2 - 16r^2s^2 + abc\left(\sum_{cyc}a\right) \geq \sum_{cyc}\left(ab\left(\sum_{cyc}a^2 - c^2\right)\right) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow 2\sum_{cyc}a^2b^2 - 16r^2s^2 + 2abc\left(\sum_{cyc}a\right) \geq \left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow 2(s^2+4Rr+r^2)^2 - 32Rrs^2 - 16r^2s^2 + 2.4Rrs.2s$$

$$\geq 2(s^2-4Rr-r^2)(s^2+4Rr+r^2) + \frac{r^4(R-2r)}{16R}$$

$$\Leftrightarrow \frac{\begin{pmatrix}32R(s^2+4Rr+r^2)^2 - 512R^2rs^2 - 256Rr^2s^2 + 256R^2rs^2\\ -32R(s^2-4Rr-r^2)(s^2+4Rr+r^2) - r^4(R-2r)\end{pmatrix}}{16R} \geq 0$$

$$\Leftrightarrow 1024R^3 + 512R^2r + 63Rr^2 + 2r^3 \overset{(*)}{\geq} 192Rs^2$$

$$\text{Now, } 192Rs^2 \overset{\text{Gerretsen}}{\leq} 192R(4R^2+4Rr+3r^2) \overset{?}{\leq} 1024R^3 + 512R^2r + 63Rr^2$$

$$+2r^3 \Leftrightarrow 256t^3 - 256t^2 - 513t + 2 \overset{?}{\geq} 0\ \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(256t^2+256t-1)\overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \boxed{\sum_{cyc} a^4 + abc(a+b+c) \ge \sum_{cyc} ab(a^2+b^2) + \frac{r^4(R-2r)}{16R}}$$

$\forall \Delta ABC, ''='' $ iff ΔABC is equilateral

$$\text{Again, } \sqrt[3]{\frac{a^3+b^3+c^3}{3}} - \frac{a+b+c}{3} = \frac{\frac{a^3+b^3+c^3}{3} - \left(\frac{a+b+c}{3}\right)^3}{\left(\frac{\sum_{cyc} a^3}{3}\right)^{\frac{2}{3}} + \left(\frac{\sum_{cyc} a}{3}\right)^2 + \left(\frac{\sum_{cyc} a^3}{3}\right)^{\frac{1}{3}}\left(\frac{\sum_{cyc} a}{3}\right)}$$

$$\le \frac{\frac{9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3}{27}}{\left(\frac{\left(\sum_{cyc} a\right)^3}{27}\right)^{\frac{2}{3}} + \left(\frac{\sum_{cyc} a}{3}\right)^2 + \left(\frac{\left(\sum_{cyc} a\right)^3}{27}\right)^{\frac{1}{3}}\left(\frac{\sum_{cyc} a}{3}\right)} = \frac{9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3}{9\left(\sum_{cyc} a\right)^2}$$

$$= \frac{18s(s^2-6Rr-3r^2)-8s^3}{9.4s^2} = \frac{1}{2}\cdot\frac{5s^2-54Rr-27r^2}{9s} \overset{\text{Mitrinovic}}{\le}$$

$$\frac{1}{2}\cdot\frac{5s^2-54Rr-27r^2}{27\sqrt{3}r} \le \frac{1}{2}\cdot\frac{5s^2-54Rr-27r^2}{46r} \left(\begin{array}{c}\because 27\sqrt{3} > 46 \text{ and} \\ 5s^2-54Rr-27r^2 \ge 0\end{array}\right)$$

$$\overset{\text{Gerretsen}}{\le} \frac{1}{2}\cdot\frac{5(4R^2+4Rr+3r^2)-54Rr-27r^2}{46r} \overset{?}{\le} \frac{R^3-4r^3}{2r^2}$$

$$\Leftrightarrow 46(R^3-4r^3) \overset{?}{\ge} r(5(4R^2+4Rr+3r^2)-54Rr-27r^2)$$

$$\Leftrightarrow 23t^3-10t^2+17t-178 \overset{?}{\ge} 0 \Leftrightarrow (t-2)(23t^2+36t+89) \overset{?}{\ge} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\ge} 2 \Rightarrow \sqrt[3]{\frac{a^3+b^3+c^3}{3}} - \frac{a+b+c}{3} \le \frac{R^3-4r^3}{2r^2}$$

$$\therefore \boxed{\frac{a+b+c}{3} + \frac{R^3-4r^3}{2r^2} \ge \sqrt[3]{\frac{a^3+b^3+c^3}{3}}} \; \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1524.

In any ΔABC and $\forall n \in \mathbb{N} : n \ge 2$, the following relationship hold :

$$\frac{h_a^n}{r_b^3 r_c + w_b^3 m_c} + \frac{w_b^n}{r_c^3 r_a + m_c^3 h_a} + \frac{m_c^n}{r_a^3 r_b + h_a^3 w_b} \ge \frac{16.3^{n-3}.r^n}{3(3R^2-8r^2)^2+3R^4-64r^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a^n}{r_b^3 r_c + w_b^3 m_c} + \frac{w_b^n}{r_c^3 r_a + m_c^3 h_a} + \frac{m_c^n}{r_a^3 r_b + h_a^3 w_b} \ge$$

$$\frac{h_a^n}{r_b^3r_c+m_b^3m_c}+\frac{h_b^n}{r_c^3r_a+m_c^3m_a}+\frac{h_c^n}{r_a^3r_b+m_a^3m_b}\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}h_a\right)^n}{3^{n-2}.\left(\sum_{cyc}r_a^3r_b+\sum_{cyc}m_a^3m_b\right)}$$

$$\overset{\text{Vasc}}{\ge}\frac{\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{3^{n-2}.\left(\frac{\left(\sum_{cyc}r_a^2\right)^2+\left(\sum_{cyc}m_a^2\right)^2}{3}\right)}\overset{\text{Bergstrom}}{\ge}\frac{\left(2rs.\frac{9}{2s}\right)^n}{3^{n-3}.\left(((4R+r)^2-2s^2)^2+\frac{9}{16}\left(\sum_{cyc}a^2\right)^2\right)}$$

$$\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Leibnitz}}}{\ge}\frac{3^{n+3}.r^n}{\left((4R+r)^2-2(16Rr-5r^2)\right)^2+\frac{9}{16}.81R^4}$$

$$=\frac{16.3^{n+3}.r^n}{16(16R^2-24Rr+11r^2)^2+729R^4}\overset{?}{\ge}\frac{16.3^{n-3}.r^n}{3(3R^2-8r^2)^2+3R^4-64r^4}$$

$$\Leftrightarrow 729\left(3\left(3R^2-8r^2\right)^2+3R^4-64r^4\right)\overset{?}{\ge}16\left(16R^2-24Rr+11r^2\right)^2+729R^4$$

$$\Leftrightarrow 17045t^4+12288t^3-119824t^2+8448t+91376\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(17045t^2+80468t+133868)+222048\right)\overset{?}{\ge}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\ge}2\therefore\frac{h_a^n}{r_b^3r_c+w_b^3m_c}+\frac{w_b^n}{r_c^3r_a+m_c^3h_a}+\frac{m_c^n}{r_a^3r_b+h_a^3w_b}$$

$$\ge\frac{16.3^{n-3}.r^n}{3(3R^2-8r^2)^2+3R^4-64r^4}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1525. In any Δ ABC with F $\to$ area, the following relationship holds :

$$F\le\frac{\sqrt{3}}{4}max\left(\frac{a^{n+2}+b^{n+2}}{a^n+b^n},\frac{b^{n+2}+c^{n+2}}{b^n+c^n},\frac{c^{n+2}+a^{n+2}}{c^n+a^n}\right),n\in\mathbb{N}^*$$

Proposed by Marian Ursărescu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$3\,max\left(\frac{a^{n+2}+b^{n+2}}{a^n+b^n},\frac{b^{n+2}+c^{n+2}}{b^n+c^n},\frac{c^{n+2}+a^{n+2}}{c^n+a^n}\right)\ge\sum_{cyc}\frac{b^{n+2}+c^{n+2}}{b^n+c^n}$$

$$=\sum_{cyc}\frac{b^n.b^2+c^n.c^2}{b^n+c^n}\overset{\text{Chebyshev}}{\ge}\sum_{cyc}\frac{\frac{1}{2}(b^n+c^n)\left(b^2+c^2\right)}{b^n+c^n}$$

$$\left(\because b\ge c\Rightarrow b^n\ge c^n\text{ and }b^2\ge c^2\text{ and }b\le c\Rightarrow b^n\le c^n\text{ and }b^2\le c^2\right)$$

$$=\sum_{cyc}a^2\overset{\text{Ionescu-Weitzenbock}}{\ge}4\sqrt{3}F$$

$$\Rightarrow \frac{\sqrt{3}}{4} max\left(\frac{a^{n+2}+b^{n+2}}{a^n+b^n}, \frac{b^{n+2}+c^{n+2}}{b^n+c^n}, \frac{c^{n+2}+a^{n+2}}{c^n+a^n}\right) \geq F$$

$$\therefore F \leq \frac{\sqrt{3}}{4} max\left(\frac{a^{n+2}+b^{n+2}}{a^n+b^n}, \frac{b^{n+2}+c^{n+2}}{b^n+c^n}, \frac{c^{n+2}+a^{n+2}}{c^n+a^n}\right) \forall \Delta ABC \text{ and } \forall n \in \mathbb{N}^*, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1526.

In any ΔABC and $\forall n \in \mathbb{N} : n \geq 2$, the following relationship holds :

$$\frac{h_a^n}{r_a^8+2r_a^3r_b^3(r_a^2+r_b^2)+r_b^8}+\frac{w_b^n}{r_b^8+2r_b^3r_c^3(r_b^2+r_c^2)+r_c^8}+\frac{m_c^n}{r_c^8+2r_c^3r_a^3(r_c^2+r_a^2)+r_a^8} \geq \frac{128.3^{n-8}.r^n}{(81R^5-2560r^5)(9R^3-64r^3)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall x,y,z>0, \sum_{cyc} x^8+\sum_{cyc} x^3y^3(x^2+y^2) = \left(\sum_{cyc} x^3\right)\left(\sum_{cyc} x^5\right) \to (1)$$

$$r_b+r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now}, \sum_{cyc} r_a^3 = \left(\sum_{cyc} r_a\right)^3 - 3(r_a+r_b)(r_b+r_c)(r_c+r_a) \overset{\text{via (i) and analogs}}{=}$$

$$(4R+r)^3 - 3.64R^3.\frac{s^2}{16R^2} \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq} \left(\frac{9R}{2}\right)^3 - 24r.27r^2 \therefore \sum_{cyc} r_a^3 \overset{(2)}{\leq} \frac{81}{8}(9R^3-64r^3)$$

$$\text{Again}, \sum_{cyc} r_a^5 \overset{\text{via (2)}}{\leq} \left(\sum_{cyc} r_a^2\right)\left(\sum_{cyc} r_a^3\right) - \sum_{cyc} r_a^2r_b^2(r_a+r_b)$$

$$= \left((4R+r)^2-2s^2\right).\frac{81}{8}(9R^3-64r^3) - (4R+r)\left(\sum_{cyc} r_a^2r_b^2\right) + r_ar_br_c\sum_{cyc} r_ar_b$$

$$\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq} \left(\left(\frac{9R}{2}\right)(4R+r)-2.27r^2\right).\frac{81}{8}(9R^3-64r^3) - \frac{(4R+r)}{3}.\left(\sum_{cyc} r_ar_b\right)^2$$

$$+rs^4 \overset{\text{Mitrinovic}}{\leq} \left(\left(\frac{9R}{2}\right)(4R+r)-2.27r^2\right).\frac{81}{8}(9R^3-64r^3) - \frac{(4R+r)}{3}.729r^4$$

$$+r.\frac{729R^4}{16}\overset{Euler}{\le}729\left(\left(\frac{R(4R+r)}{2}-6r^2\right)\left(\frac{9R^3-64r^3}{8}\right)-3r^5+\frac{R^4r}{16}\right)$$

$$=\frac{729\left(\left(R(4R+r)-12r^2\right)(9R^3-64r^3)-48r^5+R^4r\right)}{16}\overset{?}{\le}\frac{729(81R^5-2560r^5)}{32}$$

$$\Leftrightarrow 9t^5-20t^4+216t^3+512t^2+128t-4000\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(8t^4+t^3(t-2)+212t^2+936t+2000\right)\overset{?}{\ge}0\to true\because t\overset{Euler}{\ge}2$$

$$\therefore\sum_{cyc}r_a^5\overset{(3)}{\le}\frac{729(81R^5-2560r^5)}{32}$$

$$Also,\frac{h_a^n}{r_a^8+2r_a^3r_b^3(r_a^2+r_b^2)+r_b^8}+\frac{w_b^n}{r_b^8+2r_b^3r_c^3(r_b^2+r_c^2)+r_c^8}$$

$$+\frac{m_c^n}{r_c^8+2r_c^3r_a^3(r_c^2+r_a^2)+r_a^8}\ge\frac{h_a^n}{r_a^8+2r_a^3r_b^3(r_a^2+r_b^2)+r_b^8}+\frac{h_b^n}{r_b^8+2r_b^3r_c^3(r_b^2+r_c^2)+r_c^8}$$

$$+\frac{h_c^n}{r_c^8+2r_c^3r_a^3(r_c^2+r_a^2)+r_a^8}\overset{Holder}{\ge}\frac{\left(\sum_{cyc}h_a\right)^n}{3^{n-2}.\sum_{cyc}\left(r_b^8+2r_b^3r_c^3(r_b^2+r_c^2)+r_c^8\right)}$$

$$=\frac{\left(2rs\sum_{cyc}\frac{1}{a}\right)^n}{2.3^{n-2}.\left(\sum_{cyc}r_a^8+\sum_{cyc}r_a^3r_b^3(r_a^2+r_b^2)\right)}\overset{\substack{Bergstrom\\ and\\ via\ (1)}}{\ge}\frac{\left(2rs.\frac{9}{2s}\right)^n}{2.3^{n-2}.\left(\sum_{cyc}r_a^3\right)\left(\sum_{cyc}r_a^5\right)}\overset{via\ (2)\ and\ (3)}{\ge}$$

$$\frac{3^{n+2}.r^n}{2.\frac{81}{8}(9R^3-64r^3).\frac{729(81R^5-2560r^5)}{32}}=\frac{128.3^{n-8}.r^n}{(81R^5-2560r^5)(9R^3-64r^3)}$$

$\forall\,\Delta\,ABC,''='' $ iff $\Delta\,ABC$ is equilateral (QED)

1527. In any scalene $\Delta\,ABC$, the following relationship holds :

$$\frac{b+c}{(r_a-r_b)(r_a-r_c)}+\frac{a+c}{(r_b-r_a)(r_b-r_c)}+\frac{a+b}{(r_c-r_a)(r_c-r_b)}=\frac{2}{a+b+c}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution 1 by proposer

$$\sum\frac{b+c}{(r_a-r_b)(r_a-r_c)}=\sum\frac{2p-a}{p^2r^2\left(\frac{1}{p-a}-\frac{1}{p-b}\right)\left(\frac{1}{p-a}-\frac{1}{p-c}\right)}$$

$$=\frac{1}{p^2r^2}\cdot\sum\frac{(2p-a)(p-a)^2(p-b)(p-c)}{(a-b)(a-c)}=\frac{pr^2}{p^2r^2}\cdot\sum\frac{(2p-a)(p-a)}{(a-b)(a-c)}$$

$$=\frac{1}{p}\sum\frac{2p^2-3pa+a^2}{(a-b)(a-c)}=$$

$$=\frac{1}{p}\left\{2p^2\cdot\sum\underbrace{\frac{1}{(a-b)(a-c)}}_{0}-3p\cdot\sum\underbrace{\frac{a}{(a-b)(a-c)}}_{0}+\sum\underbrace{\frac{a^2}{(a-b)(a-c)}}_{1}\right\}$$

$$=\frac{1}{p}\sum\frac{a^2}{(a-b)(a-c)}=\frac{1}{p}\cdot 1=\frac{2}{2p}=\frac{2}{a+b+c}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{b+c}{(r_a-r_b)(r_a-r_c)}=\frac{b+c}{\left(\frac{rs}{s-a}-\frac{rs}{s-b}\right)\left(\frac{rs}{s-a}-\frac{rs}{s-c}\right)}$$

$$=\frac{(b+c)(s-a)^2(s-b)(s-c)}{r^2s^2(a-b)(a-c)}=\frac{r^2s(b+c)(s-a)(c-b)}{r^2s^2(a-b)(b-c)(c-a)}$$

$$=\frac{(b+c)(s-a)\big((s-b)-(s-c)\big)}{s(a-b)(b-c)(c-a)}$$

$$=\frac{1}{s(a-b)(b-c)(c-a)}\cdot\begin{pmatrix}\big(s+(s-a)\big)(s-a)(s-b)\\-\big(s+(s-a)\big)(s-a)(s-c)\end{pmatrix}$$

$$=\frac{s(s-a)(s-b)+(s-a)^2(s-b)-s(s-a)(s-c)-(s-a)^2(s-c)}{s\big((s-b)-(s-a)\big)\big((s-c)-(s-b)\big)\big((s-a)-(s-c)\big)}$$

$$=\frac{s\big((s-a)(s-b)-(s-a)(s-c)\big)+x^2y-x^2z}{s(y-x)(z-y)(x-z)}\text{ and analogs}$$

$$(x=s-a,y=s-b,z=s-c)$$

$$\therefore\frac{b+c}{(r_a-r_b)(r_a-r_c)}+\frac{a+c}{(r_b-r_a)(r_b-r_c)}+\frac{a+b}{(r_c-r_a)(r_c-r_b)}$$

$$=\frac{s\sum_{cyc}(s-a)(s-b)-s\sum_{cyc}(s-a)(s-c)+x^2y+y^2z+z^2x-xy^2-yz^2-zx^2}{s(y-x)(z-y)(x-z)}$$

$$=\frac{xy(x-y)-z(x-y)(x+y)+z^2(x-y)}{s(x-y)(y-z)(x-z)}=\frac{(x-y)(xy-zx-zy+z^2)}{s(x-y)(y-z)(x-z)}$$

$$=\frac{(x-y)\big(x(y-z)-z(y-z)\big)}{s(x-y)(y-z)(x-z)}=\frac{(x-y)(y-z)(x-z)}{s(x-y)(y-z)(x-z)}=\frac{2}{2s}=\frac{2}{a+b+c}\text{ (QED)}$$

1528. In any scalene Δ ABC, the following relationship holds :

$$\frac{(b+c)^2}{(h_a-h_b)(h_c-h_a)}+\frac{(a+c)^2}{(h_b-h_a)(h_c-h_b)}+\frac{(a+b)^2}{(h_c-h_a)(h_b-h_c)}=\frac{2R}{r}$$

Proposed by Ertan Yildirim-Izmir-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(b+c)^2}{(h_a-h_b)(h_c-h_a)}+\frac{(a+c)^2}{(h_b-h_a)(h_c-h_b)}+\frac{(a+b)^2}{(h_c-h_a)(h_b-h_c)}$$

$$=\frac{(b+c)^2}{\left(\frac{2rs}{a}-\frac{2rs}{b}\right)\left(\frac{2rs}{c}-\frac{2rs}{a}\right)}+\frac{(a+c)^2}{\left(\frac{2rs}{b}-\frac{2rs}{a}\right)\left(\frac{2rs}{c}-\frac{2rs}{b}\right)}+\frac{(a+b)^2}{\left(\frac{2rs}{c}-\frac{2rs}{a}\right)\left(\frac{2rs}{b}-\frac{2rs}{c}\right)}$$

$$=\frac{a(b+c)^2}{\frac{4r^2s^2}{4Rrs}(a-b)(c-a)}+\frac{b(a+c)^2}{\frac{4r^2s^2}{4Rrs}(a-b)(b-c)}+\frac{c(a+b)^2}{\frac{4r^2s^2}{4Rrs}(b-c)(c-a)}=$$

$$\frac{R}{rs(a-b)(b-c)(c-a)}\left(a(b-c)(b+c)^2+b(c-a)(a+c)^2+c(a-b)(a+b)^2\right)$$

$$=\frac{R\left(a(b+c)(b^2-c^2)+b(a+c)(c^2-a^2)+c(a+b)(a^2-b^2)\right)}{rs(a-b)(b-c)(c-a)}$$

$$=\frac{R\left(\left(\sum_{cyc}ab-bc\right)(b^2-c^2)+\left(\sum_{cyc}ab-ca\right)(c^2-a^2)+\left(\sum_{cyc}ab-ab\right)(a^2-b^2)\right)}{rs(a-b)(b-c)(c-a)}$$

$$=\frac{R\left(\left(\sum_{cyc}ab\right)(b^2-c^2+c^2-a^2+a^2-b^2)-\left(bc(b^2-c^2)+c^3a-ca^3+a^3b-ab^3\right)\right)}{rs(a-b)(b-c)(c-a)}$$

$$=\frac{-R}{rs(a-b)(b-c)(c-a)}\left(bc(b^2-c^2)-a(b-c)(b^2+bc+c^2)+a^3(b-c)\right)$$

$$=\frac{-R(b-c)}{rs(a-b)(b-c)(c-a)}\left(b^2c+bc^2-ab^2-abc-ac^2+a^3\right)$$

$$=\frac{-R(b-c)}{rs(a-b)(b-c)(c-a)}\left(b^2(c-a)-a(c-a)(c+a)+bc(c-a)\right)$$

$$=\frac{-R(b-c)(c-a)}{rs(a-b)(b-c)(c-a)}\left(b^2-ca-a^2+bc\right)$$

$$=\frac{-R(b-c)(c-a)}{rs(a-b)(b-c)(c-a)}\left((b+a)(b-a)+c(b-a)\right)$$

$$=\frac{-R(b-c)(c-a)(b-a)(a+b+c)}{rs(a-b)(b-c)(c-a)}=\frac{R(2s)}{rs}=\frac{2R}{r}\ \text{(QED)}$$

1529. In ΔABC the following relationship holds :

$$\frac{n_an_bn_c}{r_ar_br_c}\geq\frac{\sqrt{3}}{R}\left(max(a,b,c)-min(a,b,c)\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Norocco

We assume that $a\geq b\geq c$. Since we have

$$(a-c)^2=[(a-b)+(b-c)]^2\overset{CBS}{\leq}2[(a-b)^2+(b-c)^2],$$

then

$$\sqrt{3}\left(max(a,b,c)-min(a,b,c)\right)=\sqrt{3(a-c)^2}\leq\sqrt{2[(a-b)^2+(b-c)^2+(a-c)^2]}$$

$$=2\sqrt{s^2-3r^2-12Rr}.$$

So it suffices to prove that

$$Rn_an_bn_c \geq 2s^2r\sqrt{s^2-3r^2-12Rr} \quad (*)$$

We have

$$n_a{}^2 = s(s-a)+\frac{s(b-c)^2}{a} = s^2-\frac{s[a^2-(b-c)^2]}{a} = s^2-\frac{4s(s-b)(s-c)}{a}$$

$$= s^2-\frac{4s.sr^2}{a(s-a)} = s^2-2h_ar_a, \text{ then}$$

$$(n_an_bn_c)^2 = (s^2-2h_ar_a)(s^2-2h_br_b)(s^2-2h_cr_c)$$

$$= s^6-2s^4\sum_{cyc}h_ar_a+4r_ar_br_ch_ah_bh_c\left(s^2\sum_{cyc}\frac{1}{h_ar_a}-2\right)$$

$$= s^6-2s^4.\frac{r[s^2+(4R+r)^2]}{2R}+4s^2r.\frac{2s^2r^2}{R}\left(s^2.\frac{4R+r}{s^2r}-2\right) = \frac{s^4[(R-r)s^2-r(4R-3r)^2]}{R}.$$

So the inequality $(*)$ is equivalent to

$$R[(R-r)s^2-r(4R-3r)^2] \geq 4r^2(s^2-3r^2-12Rr)$$

$$\Leftrightarrow (R^2-Rr-4r^2)s^2 \geq 16R^3r-24R^2r^2-39Rr^3-12r^4 \quad (1)$$

▪ $If\ R^2-Rr-4r^2 \geq 0$, **we have**

$$LHS_{(1)} \overset{Rouche}{\gtreqqless} (R^2-Rr-4r^2)\left(2R^2+10Rr-r^2-2\sqrt{R(R-2r)^3}\right)$$

$$= 2R^4+8R^3r-19R^2r^2-39Rr^3+4r^4-2(R^2-Rr-4r^2)\sqrt{R(R-2r)^3}$$

$$\overset{AM-GM}{\gtreqqless} 2R^4+8R^3r-19R^2r^2-39Rr^3+4r^4-\left[(R^2-Rr-4r^2)^2+R(R-2r)^3\right] = RHS_{(1)}.$$

▪ $If\ R^2-Rr-4r^2 \leq 0$, **we have**

$$LHS_{(1)} \overset{Rouche}{\gtreqqless} (R^2-Rr-4r^2)\left(2R^2+10Rr-r^2+2\sqrt{R(R-2r)^3}\right)$$

$$= 2R^4+8R^3r-19R^2r^2-39Rr^3+4r^4-2(Rr+4r^2-R^2)\sqrt{R(R-2r)^3}$$

$$\overset{AM-GM}{\gtreqqless} 2R^4+8R^3r-19R^2r^2-39Rr^3+4r^4-\left[(Rr+4r^2-R^2)^2+R(R-2r)^3\right] = RHS_{(1)}.$$

which completes the proof.

1530.

In any Δ ABC, the following relationship holds :

$$\frac{2m_aw_a}{h_a} \geq r_b+r_c \geq \frac{m_bh_b+m_ch_c}{r_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(1)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now}, r_b + r_c - \frac{m_b h_b + m_c h_c}{r_a} \ge r_b + r_c - \frac{as}{r_a}$$

$$\left(\begin{array}{c}\frac{R}{r} \ge \frac{m_b}{h_c} + \frac{m_c}{h_b} \ldots \text{reference : article titled} \\ \text{"New Triangle Inequalities With Brocard's Angle"} \\ \text{by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 − 7,} \\ \text{published at : www.ssmrmh.ro} \therefore m_b h_b + m_c h_c \le \frac{R}{r}.h_b h_c = \frac{R}{r}.\frac{4r^2s^2.a}{bca} \\ = \frac{R}{r}.\frac{4r^2s^2.a}{4Rrs} \therefore m_b h_b + m_c h_c \le as\end{array}\right)$$

$$= \frac{\sum_{cyc} r_a r_b - r_b r_c - as}{r_a} = \frac{s^2 - s(s-a) - as}{r_a} = 0 \Rightarrow r_b + r_c - \frac{m_b h_b + m_c h_c}{r_a} \ge 0$$

$$\therefore \boxed{r_b + r_c \ge \frac{m_b h_b + m_c h_c}{r_a}}$$

$$\text{Again}, \frac{2m_a w_a}{h_a} - (r_b + r_c) \overset{\text{Lascu}}{\ge} \frac{2.\frac{b+c}{2}.\cos\frac{A}{2}.\frac{2bc}{b+c}.\cos\frac{A}{2}}{\frac{bc}{2R}} - (r_b + r_c) \overset{\text{via (1)}}{=}$$

$$4R\cos^2\frac{A}{2} - 4R\cos^2\frac{A}{2} = 0 \therefore \boxed{\frac{2m_a w_a}{h_a} \ge r_b + r_c}$$

$$\therefore \frac{2m_a w_a}{h_a} \ge r_b + r_c \ge \frac{m_b h_b + m_c h_c}{r_a} \text{ (QED)}$$

1531. In any Δ ABC, the following relationship holds :

$$(4R + r_a)(r_b + r_c - h_a) \ge n_a^2 + g_a^2 + m_b h_b + m_c h_c - r_a(h_a - 2r)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(1)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) \overset{(i)}{=} an_a^2 + a(s-b)(s-c) \text{ and}$$

$$b^2(s-b)+c^2(s-c)\overset{(ii)}{=}ag_a^2+a(s-b)(s-c)\text{ and }(i)+(i)\Rightarrow$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)$$
$$\Rightarrow 2a(b^2+c^2)=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)$$
$$\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=$$
$$2\left(n_a^2+g_a^2\right)\Rightarrow 4m_a^2+(b-c)^2=2\left(n_a^2+g_a^2\right)\Rightarrow 2\left(n_a^2+g_a^2\right)$$
$$=(b-c)^2+4s(s-a)+(b-c)^2\therefore n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$
$$\therefore(4R+r_a)(r_b+r_c-h_a)-\left(n_a^2+g_a^2+m_bh_b+m_ch_c-r_a(h_a-2r)\right)$$
$$\overset{\text{via }(*)}{=}4R(r_b+r_c)-4R.\frac{bc}{2R}+r_a(r_b+r_c)-r_ah_a-(b-c)^2-2s(s-a)$$
$$-(m_bh_b+m_ch_c)+r_ah_a-2rr_a\geq$$
$$4R(r_b+r_c)-2bc+r_a(r_b+r_c)-(b-c)^2$$
$$-\frac{1}{2}\left((b+c)^2-a^2+\frac{4(s-a)(s-b)(s-c)}{(s-a)}\right)-as$$
$$\left(\begin{gathered}\frac{R}{r}\geq\frac{m_b}{h_c}+\frac{m_c}{h_b}\ldots\text{reference : article titled}\\ \text{"New Triangle Inequalities With Brocard's Angle"}\\ \text{by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 - 7,}\\ \text{published at : www.ssmrmh.ro}\therefore m_bh_b+m_ch_c\leq\frac{R}{r}.h_bh_c=\frac{R}{r}.\frac{4r^2s^2.a}{bca}\\ =\frac{R}{r}.\frac{4r^2s^2.a}{4Rrs}\therefore m_bh_b+m_ch_c\leq as\end{gathered}\right)$$
$$=4R(r_b+r_c)+r_a(r_b+r_c)-(b^2+c^2)-\frac{1}{2}\left((b+c)^2-a^2+a^2-(b-c)^2\right)-as$$
$$\overset{\text{via }(1)}{=}4R.4R\cos^2\frac{A}{2}+\sum_{cyc}r_ar_b-r_br_c-\left(b^2+c^2+2bc\right)-as$$
$$=16R^2\cos^2\frac{A}{2}+s^2-s(s-a)-(b+c)^2-as$$
$$=16R^2\cos^2\frac{A}{2}-16R^2\cos^2\frac{A}{2}\cos^2\frac{B-C}{2}=16R^2\cos^2\frac{A}{2}\left(1-\cos^2\frac{B-C}{2}\right)\geq 0$$
$$\therefore(4R+r_a)(r_b+r_c-h_a)\geq n_a^2+g_a^2+m_bh_b+m_ch_c-r_a(h_a-2r)\ (QED)$$

1532. In ΔABC the following relationship holds:

$$\frac{\sin^2A}{bc}+\frac{\sin^2B}{ca}+\frac{\sin^2C}{ab}\geq\frac{s^2}{R^2}\cdot\frac{1}{a^2+b^2+c^2}$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Daniel Sitaru-Romania

$$\frac{\sin^2A}{bc}+\frac{\sin^2B}{ca}+\frac{\sin^2C}{ab}=\frac{\frac{a^2}{4R^2}}{bc}+\frac{\frac{b^2}{4R^2}}{ca}+\frac{\frac{c^2}{4R^2}}{ab}=$$

$$=\frac{1}{4R^2}\cdot\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)\overset{BERGSTROM}{\geq}\frac{1}{4R^2}\cdot\frac{(a+b+c)^2}{ab+bc+ca}=$$

$$=\frac{4s^2}{4R^2}\cdot\frac{1}{ab+bc+ca}\geq\frac{s^2}{R^2}\cdot\frac{1}{a^2+b^2+c^2}$$

1533. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\leq\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a^2}{r_a^2}=4\sum_{cyc}\frac{s^2-2sa+a^2}{a^2}=\frac{4s^2\sum_{cyc}a^2b^2}{16R^2r^2s^2}-\frac{8s\sum_{cyc}ab}{4Rrs}+12$$

$$\Rightarrow\sum_{cyc}\frac{h_a^2}{r_a^2}=\frac{\sum_{cyc}a^2b^2-8Rr\sum_{cyc}ab+48R^2r^2}{4R^2r^2}\rightarrow(1)$$

$$\sum_{cyc}\frac{r_a^2}{h_a^2}=\frac{1}{4}\sum_{cyc}\frac{(a-s+s)^2}{(s-a)^2}=\frac{1}{4}\left(3-2s\sum_{cyc}\frac{1}{s-a}+s^2\sum_{cyc}\frac{1}{(s-a)^2}\right)$$

$$=\frac{1}{4}\left(3-\frac{2s(4Rr+r^2)}{r^2s}+\frac{s^2}{r^4s^2}\left(\begin{array}{c}\left(\sum_{cyc}(s-b)(s-c)\right)^2\\-2(s-a)(s-b)(s-c)\sum_{cyc}(s-a)\end{array}\right)\right)$$

$$=\frac{1}{4}\left(3-\frac{2(4R+r)}{r}+\frac{r^2\left((4R+r)^2-2s^2\right)}{r^4}\right)\Rightarrow\sum_{cyc}\frac{r_a^2}{h_a^2}=\frac{8R^2+r^2-s^2}{2r^2}\rightarrow(2)$$

$$\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\sum_{cyc}\frac{\sum_{cyc}h_a^2-h_a^2}{r_a^2}=\frac{\sum_{cyc}a^2b^2}{4R^2}\cdot\frac{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}{r^2s^4}-\sum_{cyc}\frac{h_a^2}{r_a^2}$$

$$\overset{via\ (1)}{=}\frac{\sum_{cyc}a^2b^2}{4R^2}\cdot\frac{s^4-2rs^2(4R+r)}{r^2s^4}-\frac{\sum_{cyc}a^2b^2-8Rr\sum_{cyc}ab+48R^2r^2}{4R^2r^2}$$

$$=\frac{-2r^2\sum_{cyc}a^2b^2-48R^2r^2s^2+8Rr\left(s^2\sum_{cyc}ab-\sum_{cyc}a^2b^2\right)}{4R^2r^2s^2}$$

$$\Rightarrow\sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}=\frac{-s^4+\left(24R^2+4Rr-2r^2\right)s^2-r(4R+r)^3}{2R^2s^2}\rightarrow(i)$$

$$\sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}=\sum_{cyc}\frac{\sum_{cyc}r_a^2-r_a^2}{h_a^2}=\frac{\left((4R+r)^2-2s^2\right)\left(s^2-4Rr-r^2\right)}{2r^2s^2}-\sum_{cyc}\frac{r_a^2}{h_a^2}$$

$$\overset{\text{via (2)}}{=} \frac{\left((4R+r)^2-2s^2\right)(s^2-4Rr-r^2)}{2r^2s^2}-\frac{8R^2+r^2-s^2}{2r^2}$$

$$\Rightarrow \sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}=\frac{-s^4+(8R^2+16Rr+2r^2)s^2-r(4R+r)^3}{2r^2s^2}\to \text{(ii)}$$

$$\therefore \text{(i)},\text{(ii)} \Rightarrow \sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\le \sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}$$

$$\Leftrightarrow \frac{-s^4+(24R^2+4Rr-2r^2)s^2-r(4R+r)^3}{2R^2s^2}$$

$$\le \frac{-s^4+(8R^2+16Rr+2r^2)s^2-r(4R+r)^3}{2r^2s^2}$$

$$\Leftrightarrow (8R^4+16R^3r-22R^2r^2-4Rr^3+2r^4)s^2-(R^2-r^2)s^4$$

$$-r(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5)\overset{(*)}{\ge} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\ge} (8R^4+16R^3r-22R^2r^2-4Rr^3+2r^4)s^2$$

$$-(4R^2+4Rr+3r^2)(R^2-r^2)s^2$$

$$-r(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5)$$

$$=(4R^4+12R^3r-21R^2r^2+5r^4)s^2$$

$$-r(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5)$$

$$\overset{\text{Gerretsen}}{\ge} (4R^4+12R^3r-21R^2r^2+5r^4)(16Rr-5r^2)$$

$$-r(64R^5+48R^4r-52R^3r^2-47R^2r^3-12Rr^4-r^5)\overset{?}{\ge} 0$$

$$\Leftrightarrow 31t^4-86t^3+38t^2+23t-6\overset{?}{\ge} 0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(31t^2+38t+66)+135\right)\overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{h_b^2+h_c^2}{r_a^2}\le \sum_{cyc}\frac{r_b^2+r_c^2}{h_a^2}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $h_a \le \sqrt{s(s-a)}=\sqrt{r_br_c}$, then we have

$$\frac{{r_b}^2+{r_c}^2}{{h_a}^2}=\frac{{h_a}^2({r_b}^2+{r_c}^2)}{{h_a}^4}\ge \frac{{h_a}^2({r_b}^2+{r_c}^2)}{{r_b}^2{r_c}^2}=\frac{{h_a}^2}{{r_b}^2}+\frac{{h_a}^2}{{r_c}^2}\quad \text{(and analogs)}$$

Therefore

$$\sum_{cyc}\frac{{r_b}^2+{r_c}^2}{{h_a}^2}\ge \sum_{cyc}\left(\frac{{h_a}^2}{{r_b}^2}+\frac{{h_a}^2}{{r_c}^2}\right)=\sum_{cyc}\left(\frac{{h_c}^2}{{r_a}^2}+\frac{{h_b}^2}{{r_a}^2}\right)=\sum_{cyc}\frac{{h_b}^2+{h_c}^2}{{r_a}^2},$$

as desired. Equality holds iff ΔABC is equilateral.

1534. In any Δ ABC, the following relationship holds :

$$\frac{12r}{R} \le \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \le \frac{3R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

<u>Proof (1)</u>

$$\cos B+\cos C = \frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab} = \frac{bc^2+a^2b-b^3+ca^2+b^2c-c^3}{2abc}$$

$$= \frac{bc(b+c)+a^2(b+c)-(b+c)(b^2-bc+c^2)}{2abc} = \frac{(b+c)(a^2-(b-c)^2)}{2abc}$$

$$= \frac{(b+c)(4(s-b)(s-c)(s-a))}{2abc(s-a)} = \frac{2r^2s}{abc}\cdot\frac{b+c}{s-a} \Rightarrow \left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)$$

$$= \left(\frac{b^2+c^2}{bc}\right)\cdot\frac{2r^2s}{abc}\cdot\frac{b+c}{s-a} = \frac{2r^2s}{16R^2r^2s^2}\cdot\sum_{cyc}\frac{a(b+c)(b^2+c^2)}{s-a}$$

$$= \frac{1}{8R^2s}\cdot\sum_{cyc}\frac{a(s+s-a)(b^2+c^2)}{s-a} = \frac{1}{8R^2s}\cdot\left(\sum_{cyc}\frac{as(b^2+c^2)}{s-a}+\sum_{cyc}a(b^2+c^2)\right)$$

$$\overset{\text{Chebyshev}}{\le} \frac{1}{8R^2s}\cdot\left(\frac{s}{3}\left(\sum_{cyc}\frac{a}{s-a}\right)\sum_{cyc}(b^2+c^2)+\sum_{cyc}ab(2s-c)\right)$$

$$\because \text{WLOG assuming } a\ge b\ge c$$

$$\left(\Rightarrow \frac{a}{s-a}\ge\frac{b}{s-b}\ge\frac{c}{s-c} \text{ and } b^2+c^2\le c^2+a^2\le a^2+b^2\right)$$

$$= \frac{1}{8R^2s}\cdot\left(\frac{2s}{3}\left(\sum_{cyc}\frac{(a-s)+s}{s-a}\right)\left(\sum_{cyc}a^2\right)+2s(s^2+4Rr+r^2)-12Rrs\right)$$

$$\overset{\text{Leibnitz}}{\le} \frac{1}{8R^2s}\cdot\left(\frac{2s}{3}\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)(9R^2)+2s(s^2-2Rr+r^2)\right)$$

$$\overset{\text{Gerretsen}}{\le} \frac{1}{8R^2s}\cdot\left(\frac{2s(4R-2r).9R^2}{3r}+2s(4R^2+2Rr+4r^2)\right)$$

$$= \frac{9R^2(2R-r)+3r(2R^2+Rr+2r^2)}{6R^2r} \overset{?}{\le} \frac{3R}{r}$$

$$\Leftrightarrow 3r(R^2-Rr-2r^2)\overset{?}{\ge}0 \Leftrightarrow 3r(R+r)(R-2r)\overset{?}{\ge}0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \le \frac{3R}{r}$$

$$\text{Again, } \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \overset{A-G}{\geq} \sum_{cyc}\left(2(\cos B+\cos C)\right)$$

$$\left(\because \cos B+\cos C=\frac{(b+c)\left(4(s-b)(s-c)\right)}{2abc}>0 \text{ and analogs}\right)$$

$$=4\left(1+\frac{r}{R}\right) \overset{Euler}{\geq} 4\left(\frac{2r}{R}+\frac{r}{R}\right) \therefore \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \geq \frac{12r}{R}$$

$$\therefore \frac{12r}{R} \leq \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \leq \frac{3R}{r}$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

<u>**Proof (2)**</u>

$$\sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \overset{Bandila}{\leq} \sum_{cyc}\left(\left(\frac{R}{r}\right)(\cos B+\cos C)\right)$$

$$\left(\because \cos B+\cos C=\frac{(b+c)\left(4(s-b)(s-c)\right)}{2abc}>0\right)=\frac{2R}{r}\left(1+\frac{r}{R}\right) \overset{Euler}{\leq} \frac{2R}{r}\left(1+\frac{1}{2}\right)$$

$$=\frac{3R}{r}$$

$$\text{Again, } \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \overset{A-G}{\geq} \sum_{cyc}\left(2(\cos B+\cos C)\right)$$

$$\left(\because \cos B+\cos C=\frac{(b+c)\left(4(s-b)(s-c)\right)}{2abc}>0 \text{ and analogs}\right)$$

$$=4\left(1+\frac{r}{R}\right) \overset{Euler}{\geq} 4\left(\frac{2r}{R}+\frac{r}{R}\right)=\frac{12r}{R} \therefore \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \geq \frac{12r}{R}$$

$$\therefore \frac{12r}{R} \leq \sum_{cyc}\left(\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\right) \leq \frac{3R}{r}$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1535. In any ΔABC, the following relationship holds :

$$\frac{3r}{R^2} \leq \sum_{cyc} \frac{m_b+m_c}{b^2+c^2} \leq \frac{3R}{8r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_b+m_c}{b^2+c^2}=\sum_{cyc}\frac{\sum_{cyc} m_a-m_a}{b^2+c^2}=\left(\sum_{cyc} m_a\right)\left(\sum_{cyc}\frac{1}{b^2+c^2}\right)-\sum_{cyc}\frac{m_a}{b^2+c^2}$$

$$\overset{\substack{A-G,\\ \text{Tereshin}\\ \text{and}\\ \text{Leuenberger}}}{\leq}(4R+r)\left(\frac{1}{2}\sum_{cyc}\frac{a}{bca}\right)-\frac{1}{4R}\sum_{cyc}\frac{b^2+c^2}{b^2+c^2}=\frac{(4R+r)s}{4Rrs}-\frac{3}{4R}=\frac{2R-r}{2Rr}$$

$$\overset{?}{\leq}\frac{3R}{8r^2}\Leftrightarrow 3R^2-8Rr+4r^2\overset{?}{\geq}0\Leftrightarrow(3R-2r)(R-2r)\overset{?}{\geq}0\rightarrow \text{true} \because R\overset{\text{Euler}}{\geq}2r$$

$$\therefore\sum_{cyc}\frac{m_b+m_c}{b^2+c^2}\leq\frac{3R}{8r^2}$$

$$\text{Again, }\sum_{cyc}\frac{m_b+m_c}{b^2+c^2}\overset{\text{Tereshin}}{\geq}\frac{1}{4R}\sum_{cyc}\frac{c^2+a^2+a^2+b^2}{b^2+c^2}=\frac{1}{4R}\left(3+2\sum_{cyc}\frac{a^2}{b^2+c^2}\right)$$

$$\overset{\text{Nesbitt}}{\geq}\frac{3}{2R}=\frac{3r}{2Rr}\overset{\text{Euler}}{\geq}\frac{3r}{R^2}\therefore\sum_{cyc}\frac{m_b+m_c}{b^2+c^2}\geq\frac{3r}{R^2}$$

$$\therefore\frac{3r}{R^2}\leq\sum_{cyc}\frac{m_b+m_c}{b^2+c^2}\leq\frac{3R}{8r^2}\ \forall\,\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1536. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\geq\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)=\frac{3F}{2a}+\frac{F}{b}-\frac{Rp}{4c}\overset{\text{Panaitopol}}{\leq}\frac{3}{4}h_a+\frac{1}{2}h_b-\frac{m_c}{4}$$

$$\leq\frac{3}{4}m_a+\frac{1}{2}m_b-\frac{m_c}{4}\therefore\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)\leq\frac{3m_a+2m_b-m_c}{4}\rightarrow(1)$$

$$\text{Again, }\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\overset{\text{Bergstrom}}{\geq}\frac{(m_a+m_b)^2}{m_a+2m_b+m_c}\overset{?}{\geq}\frac{3m_a+2m_b-m_c}{4}$$

$$\Leftrightarrow 4(x+y)^2\overset{?}{\geq}(3x+2y-z)(x+2y+z)\ (x=m_a,y=m_b,z=m_c)$$

$$\Leftrightarrow 4x^2+4y^2+8xy\overset{?}{\geq}3x^2+6xy+3xz+2xy+4y^2+2yz-zx-2yz-z^2$$

$$\Leftrightarrow x^2-2xz+z^2\overset{?}{\geq}0\Leftrightarrow(x-z)^2\overset{?}{\geq}0\rightarrow\text{true}$$

$$\therefore\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\geq\frac{3m_a+2m_b-m_c}{4}\overset{\text{via (1)}}{\geq}\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)$$

$$\forall\,\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1537. In $a$ny Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}+\left(\frac{R}{2r}\right)^3\ge 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\overset{?}{\ge}32Rr\cos^2\frac{A}{2}\overset{\text{via (i)}}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2+4a^2-4a(b+c)\overset{?}{\ge}0$$

$$\Leftrightarrow (b+c-2a)^2\overset{?}{\ge}0\to\text{true}\ \therefore b+c\ge\sqrt{32Rr}.\cos\frac{A}{2}\ \text{and analogs}\Rightarrow$$

$$\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\le\sum_{cyc}\sqrt[2023]{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}}}=\sqrt[4046]{\frac{R}{2r}}.\sum_{cyc}\sqrt[2023]{\sin\frac{A}{2}}$$

$$\overset{\text{Jensen}}{\le}\sqrt[4046]{\frac{R}{2r}}.3.\sqrt[2023]{\frac{1}{2}}$$

$$\left(\because f''(x)=-\frac{2023\sin^2\frac{A}{2}+2022\cos^2\frac{A}{2}}{16370116\left(\sin\frac{A}{2}\right)^{\frac{4045}{2023}}}<0\text{ where }f(x)=\sqrt[2023]{\sin\frac{x}{2}}\ \forall x\in(0,\pi)\right)$$

$$\therefore \boxed{\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\le 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{R}{2r}}}\to(1)$$

$$\text{Also, }\frac{\frac{r}{R}}{\frac{9}{2}-\frac{r}{R}}\overset{?}{\ge}\frac{r^2}{2R^2}\Leftrightarrow\frac{2R}{9R-2r}\overset{?}{\ge}\frac{r}{2R}\Leftrightarrow 4R^2-9Rr+2r^2\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)(4R-r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r\ \therefore\frac{\frac{r}{R}}{\frac{9}{2}-\frac{r}{R}}\ge\frac{r^2}{2R^2}\to\text{(ii)}$$

$$\text{Again, }\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}\overset{A-G}{\geq}3.\sqrt[3]{\sqrt[2023]{\frac{\prod_{cyc}\sin\frac{A}{2}}{\prod_{cyc}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}}}$$

$$=3.\sqrt[6069]{\frac{\frac{r}{4R}}{\left(\sum_{cyc}\sin\frac{A}{2}\right)\left(\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2}\right)-\frac{r}{4R}}}\geq 3.\sqrt[6069]{\frac{\frac{r}{4R}}{\frac{\left(\sum_{cyc}\sin\frac{A}{2}\right)^3}{3}-\frac{r}{4R}}}$$

$$\overset{Jensen}{\geq}3.\sqrt[6069]{\frac{\frac{r}{4R}}{\frac{\left(\frac{3}{2}\right)^3}{3}-\frac{r}{4R}}}\overset{via\ (ii)}{\geq}3.\sqrt[6069]{\frac{r^2}{2R^2}}=3.\sqrt[6069]{\frac{4r^2}{R^2}\cdot\frac{1}{2^3}}$$

$$\therefore \boxed{\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}\geq 3.\sqrt[2023]{\frac{1}{2}}\cdot\sqrt[6069]{\frac{4r^2}{R^2}}}\to(2)\ \therefore(1),(2)\Rightarrow\text{in order to prove :}$$

$$\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}+\frac{R}{2r}\geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}},\text{ it suffices to prove :}$$

$$3.\sqrt[2023]{\frac{1}{2}}\cdot\sqrt[6069]{\frac{4r^2}{R^2}}+\frac{R}{2r}-1\geq 3.\sqrt[2023]{\frac{1}{2}}\cdot\sqrt[4046]{\frac{R}{2r}}$$

$$\Leftrightarrow\frac{R}{2r}-1\geq 3.\sqrt[2023]{\frac{1}{2}}\cdot\left(\left(\frac{R}{2r}\right)^{\frac{1}{4046}}-\left(\frac{2r}{R}\right)^{\frac{2}{6069}}\right)$$

$$\Leftrightarrow\left(\left(\frac{R}{2r}\right)^{\frac{1}{12138}}\right)^{12138}-1\geq 3.\sqrt[2023]{\frac{1}{2}}\cdot\left(\left(\left(\frac{R}{2r}\right)^{\frac{1}{12138}}\right)^3-\left(\left(\frac{2r}{R}\right)^{\frac{1}{12138}}\right)^4\right)$$

$$\Leftrightarrow\boxed{t^{12138}-1\overset{(*)}{\geq}3.\sqrt[2023]{\frac{1}{2}}\cdot\left(t^3-\frac{1}{t^4}\right)}\left(t=\left(\frac{R}{2r}\right)^{\frac{1}{12138}}\geq 1\right)$$

$$\because 2^{\frac{1}{2023}}>1\text{ and }\because t^3-\frac{1}{t^4}\geq 0\ \therefore 3.\sqrt[2023]{\frac{1}{2}}\cdot\left(t^3-\frac{1}{t^4}\right)\leq 3\left(t^3-\frac{1}{t^4}\right)\overset{?}{\leq}t^{12138}-1$$

$$\Leftrightarrow\boxed{t^{12138}-3t^3+\frac{3}{t^4}-1\overset{?}{\geq}0\overset{(**)}{\geq}0}$$

$$\text{Let }f(t)=t^{12138}-3t^3+\frac{3}{t^4}-1\ \forall\ t\geq 1\text{ and then : }f'(t)=\frac{12138t^{12142}-9t^7-12}{t^5}$$

$$=\frac{9\left(t^{12142}-t^7\right)+12\left(t^{12142}-1\right)+12117t^{12142}}{t^5}\geq\frac{12117t^{12142}}{t^5}$$

$$\left(\because t=\left(\frac{R}{2r}\right)^{\frac{1}{12138}}\overset{Euler}{\geq}1\right)>0\Rightarrow f(t)\text{ is }\uparrow\forall t\geq 1\Rightarrow f(t)\geq f(1)=0\Rightarrow(**)\Rightarrow(*)$$

$$\text{is true}\therefore\frac{R}{2r}\geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}-\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}\ and\ \because\left(\frac{R}{2r}\right)^3\geq\frac{R}{2r}\therefore$$

$$\left(\frac{R}{2r}\right)^3\geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}-\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}\therefore\sum_{cyc}\sqrt[2023]{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}+\sin\frac{C}{2}}}+\left(\frac{R}{2r}\right)^3$$

$$\geq 1+\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\ \forall\Delta ABC,''=''\ \text{iff}\ \Delta ABC\text{ is equilateral (QED)}$$

1538.

Prove that for any acute triangle *ABC* the following inequalities holds

$$\sqrt{\frac{3}{2}}\leq\frac{\sin A+\sin B+\sin C}{\sqrt{\cos A}+\sqrt{\cos B}+\sqrt{\cos C}}<2$$

Proposed by Vasile Mircea Popa-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sqrt{\cos A}+\sqrt{\cos B}+\sqrt{\cos C}\leq\sqrt{(a\cos A+b\cos B+c\cos C)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)},$$

with $a\cos A+b\cos B+c\cos C=\frac{2F}{R}$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{ab+bc+ca}{abc}\leq\frac{(a+b+c)^2}{3.4RF}$, then

$$\sqrt{\cos A}+\sqrt{\cos B}+\sqrt{\cos C}\leq\sqrt{\frac{2}{3}}\cdot\frac{a+b+c}{2R}=\sqrt{\frac{2}{3}}.(\sin A+\sin B+\sin C),$$

which completes the proof of the left side inequality. Equality holds iff ΔABC is equilateral.

Now since $\sqrt{\cos A}\geq\cos A$ (and analogs),

then to prove the right side inequality it suffices to prove that

$$\sin A+\sin B+\sin C<2(\cos A+\cos B+\cos C)\ \text{or}\ \frac{s}{R}<2\left(1+\frac{r}{R}\right)\ \text{or}\ s<2(R+r)$$

which is true by Gerretsen's inequality, $s\leq\sqrt{4R^2+4Rr+3r^2}<2(R+r)$.

1539.

In any ΔABC the following relationship holds :

$$1.\ \frac{h_aw_b}{h_a+w_b+2m_c}+\frac{w_bm_c}{w_b+m_c+2h_a}+\frac{m_ch_a}{m_c+h_a+2w_b}\leq\frac{9R}{8}$$

2. $\frac{27r^3}{R} \le \frac{h_a{w_b}^2}{h_a+w_b}+\frac{w_b{m_c}^2}{w_b+m_c}+\frac{m_c{h_a}^2}{m_c+h_a} \le \frac{27R^2}{8}$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

1. By CBS inequality, we have

$$\frac{h_aw_b}{h_a+w_b+2m_c}+\frac{w_bm_c}{w_b+m_c+2h_a}+\frac{m_ch_a}{m_c+h_a+2w_b}$$
$$\le \frac{1}{4}\left(\frac{h_aw_b}{h_a+m_c}+\frac{h_aw_b}{w_b+m_c}\right)+\frac{1}{4}\left(\frac{w_bm_c}{w_b+h_a}+\frac{w_bm_c}{m_c+h_a}\right)+\frac{1}{4}\left(\frac{m_ch_a}{m_c+w_b}+\frac{m_ch_a}{h_a+w_b}\right)$$
$$=\frac{h_a+w_b+m_c}{4} \overset{h_a\le m_a\,\&\,w_b\le m_b}{\le} \frac{m_a+m_b+m_c}{4} \overset{Gotman}{\le} \frac{1}{4}.\frac{9R}{2}=\frac{9R}{8},$$

as desired. Equality holds iff ΔABC is equilateral.

2. By AM – GM inequality, we have

$$\frac{h_a{w_b}^2}{h_a+w_b}+\frac{w_b{m_c}^2}{w_b+m_c}+\frac{m_c{h_a}^2}{m_c+h_a} \le \frac{w_b(h_a+w_b)}{4}+\frac{m_c(w_b+m_c)}{4}+\frac{h_a(m_c+h_a)}{4}$$
$$=\frac{(h_aw_b+w_bm_c+m_ch_a)+({h_a}^2+{w_b}^2+{m_c}^2)}{4} \le \frac{{h_a}^2+{w_b}^2+{m_c}^2}{2}$$
$$\overset{h_a\le m_a\,\&\,w_b\le m_b}{\le} \frac{{m_a}^2+{m_b}^2+{m_c}^2}{2}=\frac{3(a^2+b^2+c^2)}{8} \overset{Leibniz}{\le} \frac{3.9R^2}{8}=\frac{27R^2}{8}.$$

Now, by using AM – GM inequality and $h_a \le w_a \le m_a$ (and analogs), we have

$$\frac{h_a{w_b}^2}{h_a+w_b}+\frac{w_b{m_c}^2}{w_b+m_c}+\frac{m_c{h_a}^2}{m_c+h_a} \ge \frac{3h_aw_bm_c}{\sqrt[3]{(h_a+w_b)(w_b+m_c)(m_c+h_a)}} \ge \frac{9h_aw_bm_c}{2(h_a+w_b+m_c)}$$
$$\ge \frac{9h_ah_bh_c}{2(m_a+m_b+m_c)} \overset{GM-HM\,\&\,Gotman}{\ge} \frac{9.\left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^3}{9R}=\frac{27r^3}{R}.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.

1540. In any Δ ABC and $\forall\, n \in \mathbb{N} : n \ge m +$

1, the following relationship holds :

$$\frac{h_a^n}{(r_a^3+h_a^3)^m}+\frac{w_b^n}{(r_b^3+w_b^3)^m}+\frac{m_c^n}{(r_c^3+m_c^3)^m} \ge \frac{2^{2m}.3^{n-3m+1}.r^n}{(9R^3-64r^3)^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} r_a^3 + \sum_{cyc} m_a^3 = \left(\sum_{cyc} r_a\right)^3 - 3(r_a+r_b)(r_b+r_c)(r_c+r_a) +$$

$$\left(\sum_{cyc} m_a\right)^3 - 3(m_a+m_b)(m_b+m_c)(m_c+m_a) \overset{\text{Leuenberger}}{\le} 2(4R+r)^3$$

$$-3(r_a+r_b)(r_b+r_c)(r_c+r_a) - 3(h_a+h_b)(h_b+h_c)(h_c+h_a) \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Cesaro}}}{\le}$$

$$2\left(\frac{9R}{2}\right)^3 - 24r_ar_br_c - 24h_ah_bh_c = 2\left(\frac{9R}{2}\right)^3 - 24rs^2 - 24.\frac{2r^2s^2}{R}$$

$$\overset{\substack{\text{Gerretsen + Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\le} 2\left(\frac{9R}{2}\right)^3 - 24r.27r^2 - 24.\frac{r^2.27Rr}{R} = 2.\frac{81(9R^3-64r^3)}{8}$$

$$\Rightarrow \sum_{cyc} r_a^3 + \sum_{cyc} m_a^3 \le \frac{81(9R^3-64r^3)}{4} \to (1)$$

$$\text{Also,} \left(\frac{\sum_{cyc} h_a^{\frac{n}{m+1}}}{3}\right)^{\frac{m+1}{n}} \overset{\text{Power-Mean inequality}}{\ge} \left(\frac{\sum_{cyc} h_a^1}{3}\right)^1 \left(\because \frac{n}{m+1} \ge 1\right) \Rightarrow \frac{\sum_{cyc} h_a^{\frac{n}{m+1}}}{3}$$

$$\ge \left(\frac{\sum_{cyc} h_a}{3}\right)^{\frac{n}{m+1}} = \left(\frac{2rs\sum_{cyc}\frac{1}{a}}{3}\right)^{\frac{n}{m+1}} \overset{\text{Bergstrom}}{\ge} \left(\frac{2rs.\frac{9}{2s}}{3}\right)^{\frac{n}{m+1}} = (3r)^{\frac{n}{m+1}}$$

$$\therefore \sum_{cyc} h_a^{\frac{n}{m+1}} \ge 3(3r)^{\frac{n}{m+1}} \to (2)$$

$$\text{Now,} \frac{h_a^n}{(r_a^3+h_a^3)^m} + \frac{w_b^n}{(r_b^3+w_b^3)^m} + \frac{m_c^n}{(r_c^3+m_c^3)^m} \ge$$

$$\frac{h_a^n}{(r_a^3+m_a^3)^m} + \frac{h_b^n}{(r_b^3+m_b^3)^m} + \frac{h_c^n}{(r_c^3+m_c^3)^m} = \sum_{cyc} \frac{\left(h_a^{\frac{n}{m+1}}\right)^{m+1}}{(r_a^3+m_a^3)^m} \overset{\text{Radon}}{\ge}$$

$$\frac{\left(\sum_{cyc} h_a^{\frac{n}{m+1}}\right)^{m+1}}{\left(\sum_{cyc} r_a^3 + \sum_{cyc} m_a^3\right)^m} \overset{\text{via (1) and (2)}}{\ge} \frac{\left(3(3r)^{\frac{n}{m+1}}\right)^{m+1}}{\left(\frac{81(9R^3-64r^3)}{4}\right)^m} = \frac{2^{2m}.3^{m+1+n-4m}.r^n}{(9R^3-64r^3)^m}$$

$$\therefore \frac{h_a^n}{(r_a^3+h_a^3)^m} + \frac{w_b^n}{(r_b^3+w_b^3)^m} + \frac{m_c^n}{(r_c^3+m_c^3)^m} \ge \frac{2^{2m}.3^{n-3m+1}.r^n}{(9R^3-64r^3)^m}$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1541.

In any Δ ABC and $\forall$ m, n $\in \mathbb{N}$: n $\geq$ m + 1, the following relationship holds :

$$\frac{h_a^n}{((r_a^3+r_b^3)^2+(h_a^3+h_b^3)^2)^m}+\frac{w_b^n}{((r_b^3+r_c^3)^2+(w_b^3+w_c^3)^2)^m}+\frac{m_c^n}{((r_c^3+r_a^3)^2+(m_c^3+m_a^3)^2)^m}\geq\frac{2^{3m}.3^{n-6m+1}.r^n}{(3(9R^3-64r^3)^2-128r^6)^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(r_b^3+r_c^3\right)^2=\left(\sum_{cyc}\left(r_b^3+r_c^3\right)\right)^2-$$

$$2\left(\left(r_b^3+r_c^3\right)\left(r_c^3+r_a^3\right)+\left(r_c^3+r_a^3\right)\left(r_a^3+r_b^3\right)+\left(r_a^3+r_b^3\right)\left(r_b^3+r_c^3\right)\right)\overset{A-G}{\leq}$$

$$4\left(\left(\sum_{cyc}r_a\right)^3-3(r_a+r_b)(r_b+r_c)(r_c+r_a)\right)^2$$

$$-6\sqrt[3]{\left(r_a^3+r_b^3\right)^2\left(r_b^3+r_c^3\right)^2\left(r_c^3+r_a^3\right)^2}\overset{\substack{\text{Leuenberger + Euler}\\ and\\ \text{Cesaro}}}{\leq}4\left(\left(\frac{9R}{2}\right)^3-24r_ar_br_c\right)^2$$

$$-6.\sqrt[3]{64(r_ar_br_c)^6}\overset{\text{Mitrinovic}}{\leq}4\left(\left(\frac{9R}{2}\right)^3-24.27r^3\right)^2-24\left(27r^3\right)^2$$

$$=4.\frac{81^2\left(9R^3-64r^3\right)^2}{64}-729.24r^6$$

$$\therefore\sum_{cyc}\left(r_b^3+r_c^3\right)^2\leq\frac{81^2\left(9R^3-64r^3\right)^2}{16}-729.24r^6\to(1)$$

$$\text{Again,}\sum_{cyc}\left(m_b^3+m_c^3\right)^2=\left(\sum_{cyc}\left(m_b^3+m_c^3\right)\right)^2-$$

$$2\left(\left(m_b^3+m_c^3\right)\left(m_c^3+m_a^3\right)+\left(m_c^3+m_a^3\right)\left(m_a^3+m_b^3\right)+\left(m_a^3+m_b^3\right)\left(m_b^3+m_c^3\right)\right)\overset{A-G}{\leq}$$

$$4\left(\left(\sum_{cyc}m_a\right)^3-3(m_a+m_b)(m_b+m_c)(m_c+m_a)\right)^2$$

$$-6\sqrt[3]{\left(m_a^3+m_b^3\right)^2\left(m_b^3+m_c^3\right)^2\left(m_c^3+m_a^3\right)^2}\overset{\substack{\text{Leuenberger + Euler}\\ and\\ \text{Cesaro}}}{\leq}$$

$$4\left(\left(\frac{9R}{2}\right)^3-24m_am_bm_c\right)^2-6.\sqrt[3]{64(m_am_bm_c)^6}$$

$$\le 4\left(\left(\frac{9R}{2}\right)^3-24h_ah_bh_c\right)^2-6.\sqrt[3]{64(h_ah_bh_c)^6}$$

$$=4\left(\left(\frac{9R}{2}\right)^3-\frac{24r^2}{R}.2s^2\right)^2-24\left(\frac{r^2}{R}.2s^2\right)^2\overset{\text{Gerretsen + Euler}}{\le}$$

$$4\left(\left(\frac{9R}{2}\right)^3-\frac{24r^2}{R}.27Rr\right)^2-24\left(\frac{r^2}{R}.27Rr\right)^2=4.\frac{81^2(9R^3-64r^3)^2}{64}-729.24r^6$$

$$\therefore\sum_{cyc}(m_b^3+m_c^3)^2\le\frac{81^2(9R^3-64r^3)^2}{16}-729.24r^6\to(2)$$

$$\text{Also,}\left(\frac{\sum_{cyc}h_a^{\frac{n}{m+1}}}{3}\right)^{\frac{m+1}{n}}\overset{\text{Power}-Mean\text{ inequality}}{\ge}\left(\frac{\sum_{cyc}h_a^1}{3}\right)^1\left(\because\frac{n}{m+1}\ge1\right)$$

$$\Rightarrow\frac{\sum_{cyc}h_a^{\frac{n}{m+1}}}{3}\ge\left(\frac{\sum_{cyc}h_a}{3}\right)^{\frac{n}{m+1}}=\left(\frac{2rs\sum_{cyc}\frac{1}{a}}{3}\right)^{\frac{n}{m+1}}\overset{\text{Bergstrom}}{\ge}\left(\frac{2rs.\frac{9}{2s}}{3}\right)^{\frac{n}{m+1}}=(3r)^{\frac{n}{m+1}}$$

$$\therefore\sum_{cyc}h_a^{\frac{n}{m+1}}\ge3(3r)^{\frac{n}{m+1}}\to(3)$$

$$\text{Now,}\frac{h_a^n}{\left((r_a^3+r_b^3)^2+(h_a^3+h_b^3)^2\right)^m}+\frac{w_b^n}{\left((r_b^3+r_c^3)^2+(w_b^3+w_c^3)^2\right)^m}$$

$$+\frac{m_c^n}{\left((r_c^3+r_a^3)^2+(m_c^3+m_a^3)^2\right)^m}\ge\frac{h_a^n}{\left((r_a^3+r_b^3)^2+(m_a^3+m_b^3)^2\right)^m}+$$

$$\frac{h_b^n}{\left((r_b^3+r_c^3)^2+(m_b^3+m_c^3)^2\right)^m}+\frac{h_c^n}{\left((r_c^3+r_a^3)^2+(m_c^3+m_a^3)^2\right)^m}$$

$$=\sum_{cyc}\frac{\left(h_a^{\frac{n}{m+1}}\right)^{m+1}}{\left((r_b^3+r_c^3)^2+(m_b^3+m_c^3)^2\right)^m}\overset{Radon}{\ge}\frac{\left(\sum_{cyc}h_a^{\frac{n}{m+1}}\right)^{m+1}}{\left(\sum_{cyc}(r_b^3+r_c^3)^2+\sum_{cyc}(m_b^3+m_c^3)^2\right)^m}$$

$$\overset{via\ (1),(2)\ and\ (3)}{\ge}\frac{\left(3(3r)^{\frac{n}{m+1}}\right)^{m+1}}{\left(\frac{81^2(9R^3-64r^3)^2}{16}-729.24r^6+\frac{81^2(9R^3-64r^3)^2}{16}-729.24r^6\right)^m}$$

$$=\frac{3^{m+1+n}.r^n}{3^7\left(\frac{3(9R^3-64r^3)^2}{8}-16r^6\right)^m}=\frac{2^{3m}.3^{m+1+n-7m}.r^n}{(3(9R^3-64r^3)^2-128r^6)^m}$$

$$\therefore\frac{h_a^n}{\left((r_a^3+r_b^3)^2+(h_a^3+h_b^3)^2\right)^m}+\frac{w_b^n}{\left((r_b^3+r_c^3)^2+(w_b^3+w_c^3)^2\right)^m}$$

$$+\frac{m_c^n}{\left(\left(r_c^3+r_a^3\right)^2+\left(m_c^3+m_a^3\right)^2\right)^m}\geq\frac{2^{3m}.3^{n-6m+1}.r^n}{(3(9R^3-64r^3)^2-128r^6)^m}$$

$\forall\,\Delta\,ABC$ and $\forall\, m,n\in\mathbb{N}: n\geq m+1,'' ='' $ iff $\Delta\,ABC$ is equilateral (QED)

1542.

In any $\Delta\,ABC$ and $\forall\, m,n,t\in\mathbb{N}: n\geq m+1$, the following relationship holds:

$$\frac{((w_a+w_b)^t+(m_a+m_b)^t)^n}{\left(w_a^5+w_a^2w_b^3+w_b^5+m_a^5+m_a^2m_b^3+m_b^5\right)^m}$$
$$+\frac{((w_b+w_c)^t+(m_b+m_c)^t)^n}{\left(w_b^5+w_b^2w_c^3+w_c^5+m_b^5+m_b^2m_c^3+m_c^5\right)^m}$$
$$+\frac{((w_c+w_a)^t+(m_c+m_a)^t)^n}{(w_c^5+w_c^2w_a^3+w_a^5+m_c^5+m_c^2m_a^3+m_a^5)^m}\geq\frac{2^{4m+n(t+1)}.3^{nt-6m+1}.r^{nt}}{(81R^5-2560r^5)^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Tapas Das-India

Note:

$$(x+y+z)^5=x^5+y^5+z^5+5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$$

$$m_a+m_b+m_c\overset{Leunberger}{\leq}4R+r\overset{Euler}{\leq}\frac{9R}{2}$$

$$(m_a+m_b)(m_b+m_c)(m_c+m_a)\overset{AM-GM}{\geq}8m_am_bm_c$$

$$\left(m_a\geq\sqrt{s(s-a)}\right)\geq 8s^2r\overset{Mitrinovic}{\geq}8\cdot 3^3r^3$$

analog

$$\sum m_a^2+\sum m_bm_c\geq 2\sum m_bm_c$$

$$\left(\because\sum m_a^2\geq\sum m_bm_c\right)$$

$$(\because m_a\geq h_a)\geq 2\sum h_bh_c=\frac{4s^2r}{R}\geq 2\cdot 3^3\cdot r^2$$

$$\therefore\sum m_a^5=\left(\sum m_a\right)^5-5\pi(m_a+m_b)\cdot\left(\sum m_a^2+\sum m_bm_c\right)$$

$$\leq\left(\frac{9R}{2}\right)^5-5\cdot 8\cdot 3^3\cdot r^3\cdot 2\cdot 3^3\cdot r^2=\frac{36(81R^5-2560r^5)}{32}$$

Applying AM-GM $h_ah_bh_c\geq 27r^3$

$$\therefore h_a+h_b+h_c\ge 3(h_ah_bh_c)^{\frac{1}{3}}=9r$$

$$w_a^5+w_a^2w_b^3+w_b^5+m_a^5+m_a^2m_b^3+m_b^5\le\le m_a^5+m_a^2m_b^3+m_b^5+m_a^5+m_a^2m_b^3+m_b^5$$

($w_a\le m_a$ analog)

$$=2(m_a^5+m_b^5)+m_a^2m_b^2(m_a+m_b)\le 2(m_a^5+m_b^5)+m_a^5+m_b^5=3(m_a^5+m_b^5)$$

Note:

$$\left[m_a^5+m_b^5\overset{CBS}{\ge}\frac{(m_a^4+m_b^4)(m_a+m_b)}{2}\right]\overset{AM-GM}{\ge}m_a^2m_b^2(m_a+m_b)$$

Note:

$$(w_a+w_b)^t+(m_a+m_b)^t\ge(h_a+h_b)^t+(h_a+h_b)^t\qquad\left(\because\begin{matrix}w_a\ge h_a\\ m_a\ge h_a\end{matrix}\right)$$

$$=2(h_a+h_b)^t$$

$$\therefore[(w_a+w_b)^t+(m_a+m_b)^t]^n\ge 2^n(h_a+h_b)^{tn}\overset{AM-GM}{\ge}2^{n+tn}(h_ah_b)^{\frac{tn}{2}}$$

$$\therefore LHS\ge\sum\frac{2^{n+tn}(h_ah_b)^{\frac{tn}{2}}}{3^m(m_a^5+m_b^5)^m}\overset{AM-GM}{\ge}\frac{3\cdot 2^{n+tn}}{3^m}\frac{[(h_ah_b)\cdot(h_bh_c)\cdot(h_ch_a)]^{\frac{tn}{6}}}{[\prod(m_a^5+m_b^5)]^{\frac{m}{3}}}$$

$$\overset{AM-GM}{\ge}\frac{3\cdot 2^{n+tn}}{3^m}\frac{(h_ah_bh_c)^{\frac{tn}{3}}}{\left[\frac{2(\sum m_a^5)}{3}\right]^m}\ge 3\cdot 2^{n+tn-m}\frac{(27r^3)^{\frac{tn}{3}}}{\left[\frac{36}{2^5}(81R^5-2560r^5)\right]^m}$$

$$=\frac{3\cdot 2^{n+tn-m+5m}\cdot 3^{tn}\cdot r^{tn}}{3^{6m}(81R^5-2560r^5)^m}=\frac{2^{4m+n(t+1)}\cdot 3^{nt-6m+1}\cdot r^{nt}}{(81R^5-2560r^5)^m}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0,\sum_{cyc}x^5+\sum_{cyc}x^2y^2(x+y)=\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x^2\right)\to(1)$$

$$\text{Now},2\sum_{cyc}m_a^5+\sum_{cyc}m_a^2m_b^3$$

$$=\left(2\sum_{cyc}m_a^5+2\sum_{cyc}m_a^2m_b^3+2\sum_{cyc}m_a^3m_b^2\right)-\sum_{cyc}m_a^2m_b^3-2\sum_{cyc}m_a^3m_b^2\overset{\text{via }(1)}{=}$$

$$2\left(\sum_{cyc}m_a^3\right)\left(\sum_{cyc}m_a^2\right)-\sum_{cyc}m_a^3(m_b^2+m_c^2)-\sum_{cyc}m_a^3m_b^2$$

$$\overset{A-G}{\le} 2\left(\sum_{cyc} m_a^3\right)\left(\sum_{cyc} m_a^2\right) - 2m_a m_b m_c\left(\sum_{cyc} m_a^2\right) - 3m_a m_b m_c \cdot \sqrt[3]{(m_a m_b m_c)^2}$$

$$= 2\left(\sum_{cyc} m_a^2\right)\left(\left(\sum_{cyc} m_a\right)^3 - 3(m_a+m_b)(m_b+m_c)(m_c+m_a) - m_a m_b m_c\right)$$

$$-3m_a m_b m_c \cdot \sqrt[3]{(m_a m_b m_c)^2} \overset{\substack{A-G,\\ Cesaro,\\ Leibnitz\\ and\\ Leuenberger}}{\le} 2.\frac{3}{4}.9R^2.\left(\left(\frac{9R}{2}\right)^3 - 25m_a m_b m_c\right)$$

$$-3m_a m_b m_c \cdot \sqrt[3]{(m_a m_b m_c)^2} \le 2.\frac{3}{4}.9R^2.\left(\left(\frac{9R}{2}\right)^3 - 25.27r^3\right) - 3.27r^3.\sqrt[3]{(27r^3)^2}$$

$$\left(\because m_a m_b m_c \ge h_a h_b h_c = \frac{r^2.2s^2}{R} \overset{Gerretsen + Euler}{\ge} \frac{r^2.27Rr}{R} = 27r^3\right)$$

$$= 729\left(\frac{27R^5 - 200R^2r^3 - 16r^5}{16}\right)$$

$$\therefore \boxed{2\sum_{cyc} m_a^5 + \sum_{cyc} m_a^2 m_b^3 \le 729\left(\frac{27R^5 - 200R^2r^3 - 16r^5}{16}\right)} \to (2)$$

$$\text{Also,} \left(\frac{\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}}}{3}\right)^{\frac{m+1}{tn}} \overset{Power-Mean\ inequality}{\ge} \left(\frac{\sum_{cyc}(h_a+h_b)^1}{3}\right)^1$$

$$\left(\because \frac{tn}{m+1} \ge 1\right) \Rightarrow \frac{\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}}}{3} \ge \left(\frac{\sum_{cyc}(h_a+h_b)}{3}\right)^{\frac{tn}{m+1}}$$

$$= 2^{\frac{tn}{m+1}}.\left(\frac{2rs\sum_{cyc}\frac{1}{a}}{3}\right)^{\frac{tn}{m+1}} \overset{Bergstrom}{\ge} 2^{\frac{tn}{m+1}}.\left(\frac{2rs.\frac{9}{2s}}{3}\right)^{\frac{tn}{m+1}} = 2^{\frac{tn}{m+1}}.(3r)^{\frac{tn}{m+1}}$$

$$\therefore \boxed{\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}} \ge 3.2^{\frac{tn}{m+1}}.(3r)^{\frac{tn}{m+1}}} \to (3)$$

$$\text{We have}: \frac{((w_a+w_b)^t + (m_a+m_b)^t)^n}{(w_a^5 + w_a^2w_b^3 + w_b^5 + m_a^5 + m_a^2m_b^3 + m_b^5)^m}$$

$$+\frac{((w_b+w_c)^t + (m_b+m_c)^t)^n}{(w_b^5 + w_b^2w_c^3 + w_c^5 + m_b^5 + m_b^2m_c^3 + m_c^5)^m}$$

$$+\frac{((w_c+w_a)^t + (m_c+m_a)^t)^n}{(w_c^5 + w_c^2w_a^3 + w_a^5 + m_c^5 + m_c^2m_a^3 + m_a^5)^m}$$

$$\ge \frac{((h_a+h_b)^t + (h_a+h_b)^t)^n}{(m_a^5 + m_a^2m_b^3 + m_b^5 + m_a^5 + m_a^2m_b^3 + m_b^5)^m}$$

$$+\frac{((h_b+h_c)^t+(h_b+h_c)^t)^n}{\left(m_b^5+m_b^2m_c^3+m_c^5+m_b^5+m_b^2m_c^3+m_c^5\right)^m}$$
$$+\frac{((h_c+h_a)^t+(h_c+h_a)^t)^n}{\left(m_c^5+m_c^2m_a^3+m_a^5+m_c^5+m_c^2m_a^3+m_a^5\right)^m}$$
$$=2^n.\sum_{cyc}\frac{(h_a+h_b)^{nt}}{\left(m_b^5+m_b^2m_c^3+m_c^5+m_b^5+m_b^2m_c^3+m_c^5\right)^m}$$
$$=2^n.\sum_{cyc}\frac{\left((h_a+h_b)^{\frac{nt}{m+1}}\right)^{m+1}}{\left(m_b^5+m_b^2m_c^3+m_c^5+m_b^5+m_b^2m_c^3+m_c^5\right)^m}\overset{Radon}{\geq}2^n.\frac{\left(\sum_{cyc}(h_a+h_b)^{\frac{nt}{m+1}}\right)^{m+1}}{\left(4\sum_{cyc}m_a^5+2\sum_{cyc}m_a^2m_b^3\right)^m}$$
$$\overset{via\ (3)}{\geq}\frac{2^n.3^{m+1}.2^{tn}.(3r)^{tn}}{2^m.\left(2\sum_{cyc}m_a^5+\sum_{cyc}m_a^2m_b^3\right)^m}\overset{via\ (2)}{\geq}\frac{2^n.3^{m+1}.2^{tn}.(3r)^{tn}}{2^m.\left(729\left(\frac{27R^5-200R^2r^3-16r^5}{16}\right)\right)^m}$$
$$=\frac{2^{n+nt-m+4m}.3^{m+1+nt-6m}.r^{nt}}{(27R^5-200R^2r^3-16r^5)^m}=\frac{2^{n+nt+3m}.3^{1+nt-5m}.r^{nt}}{(27R^5-200R^2r^3-16r^5)^m}$$
$$\overset{?}{\geq}\frac{2^{4m+n(t+1)}.3^{nt-6m+1}.r^{nt}}{(81R^5-2560r^5)^m}$$
$$\Leftrightarrow\left(\frac{3}{2}\right)^m.\frac{1}{(27R^5-200R^2r^3-16r^5)^m}\overset{?}{\geq}\frac{1}{(81R^5-2560r^5)^m}$$
$$\Leftrightarrow 3\left(81R^5-2560r^5\right)\overset{?}{\geq}2\left(27R^5-200R^2r^3-16r^5\right)$$
$$\Leftrightarrow\boxed{189t^5+400t^2-7648\overset{?}{\geq}0}\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow 189\left(t^5-32\right)+400\left(t^2-4\right)\overset{?}{\geq}0\rightarrow true\because t\overset{Euler}{\geq}2$$
$$\therefore\frac{((w_a+w_b)^t+(m_a+m_b)^t)^n}{\left(w_a^5+w_a^2w_b^3+w_b^5+m_a^5+m_a^2m_b^3+m_b^5\right)^m}$$
$$+\frac{((w_b+w_c)^t+(m_b+m_c)^t)^n}{\left(w_b^5+w_b^2w_c^3+w_c^5+m_b^5+m_b^2m_c^3+m_c^5\right)^m}$$
$$+\frac{((w_c+w_a)^t+(m_c+m_a)^t)^n}{\left(w_c^5+w_c^2w_a^3+w_a^5+m_c^5+m_c^2m_a^3+m_a^5\right)^m}\geq\frac{2^{4m+n(t+1)}.3^{nt-6m+1}.r^{nt}}{(81R^5-2560r^5)^m}$$
$$\forall\,\Delta\,ABC\text{ and }\forall\,m,n,t\in\mathbb{N}:n\geq m+1,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1543.

In any $\Delta\,ABC$ and $\forall\,m,n,t\in\mathbb{N}:n\geq m+1$, the following relationship holds :

$$\frac{(h_a^t+w_b^t+m_c^t)^n}{(r_a^7+2r_a^3r_b^3(r_a+r_b)+r_b^7)^m}+\frac{(h_b^t+w_c^t+m_a^t)^n}{(r_b^7+2r_b^3r_c^3(r_b+r_c)+r_c^7)^m}$$
$$+\frac{(h_c^t+w_a^t+m_b^t)^n}{(r_c^7+2r_c^3r_a^3(r_c+r_a)+r_a^7)^m}\geq\frac{2^{6m}.3^{n(t+1)-8m+1}.r^{nt}}{\left((27R^4-416r^4)(9R^3-64r^3)\right)^m}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y,z>0, \sum_{cyc} x^7 + \sum_{cyc} x^3y^3(x+y) = \left(\sum_{cyc} x^4\right)\left(\sum_{cyc} x^3\right) \rightarrow (1)$$

$$\text{Now}, \sum_{cyc} r_a^4 = \left(\sum_{cyc} r_a^2\right)^2 - 2\sum_{cyc} r_a^2 r_b^2 \leq \left((4R+r)^2 - 2s^2\right)^2 - \frac{2}{3}\left(\sum_{cyc} r_a r_b\right)^2$$

$$\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\leq} \left(\left(\frac{9R}{2}\right)^2 - 2.27r^2\right) - \frac{2}{3}.729r^4 = \frac{243}{16}\left(3\left(9R^4 + 64r^4 - 48R^2r^2\right) - 32r^4\right)$$

$$\overset{\text{Euler}}{\leq} \frac{243}{16}\left(3\left(9R^4 + 64r^4 - 48.4r^2.r^2\right) - 32r^4\right)$$

$$\therefore \sum_{cyc} r_a^4 \leq \frac{243}{16}\left(27R^4 - 416r^4\right) \rightarrow (2)$$

$$\text{Again}, \sum_{cyc} r_a^3 = (4R+r)^3 - 3(r_a+r_b)(r_b+r_c)(r_c+r_a) \overset{\substack{\text{Euler}\\ \text{and}\\ \text{Cesaro}}}{\leq}$$

$$\left(\frac{9R}{2}\right)^3 - 24r_a r_b r_c \overset{\text{Mitrinovic}}{\leq} \left(\frac{9R}{2}\right)^3 - 24.r.27r^2 \therefore \sum_{cyc} r_a^3 \leq \frac{81}{8}\left(9R^3 - 64r^3\right) \rightarrow (3)$$

$$\text{We have}: \frac{\left(h_a^t + w_b^t + m_c^t\right)^n}{\left(r_a^7 + 2r_a^3r_b^3(r_a+r_b) + r_b^7\right)^m} + \frac{\left(h_b^t + w_c^t + m_a^t\right)^n}{\left(r_b^7 + 2r_b^3r_c^3(r_b+r_c) + r_c^7\right)^m}$$

$$+ \frac{\left(h_c^t + w_a^t + m_b^t\right)^n}{\left(r_c^7 + 2r_c^3r_a^3(r_c+r_a) + r_a^7\right)^m} = \frac{\left(\left(h_a^t + w_b^t + m_c^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{\left(r_a^7 + 2r_a^3r_b^3(r_a+r_b) + r_b^7\right)^m} + \frac{\left(\left(h_b^t + w_c^t + m_a^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{\left(r_b^7 + 2r_b^3r_c^3(r_b+r_c) + r_c^7\right)^m}$$

$$+ \frac{\left(\left(h_c^t + w_a^t + m_b^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{\left(r_c^7 + 2r_c^3r_a^3(r_c+r_a) + r_a^7\right)^m} \overset{\text{Radon}}{\geq}$$

$$\frac{\left(\left(h_a^t + w_b^t + m_c^t\right)^{\frac{n}{m+1}} + \left(h_b^t + w_c^t + m_a^t\right)^{\frac{n}{m+1}} + \left(h_c^t + w_a^t + m_b^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{\left(2\sum_{cyc} r_a^7 + 2\sum_{cyc} r_a^3 r_b^3(r_a+r_b)\right)^m}$$

$$\overset{\text{via }(1)}{\geq} \frac{\left(3\left(\sum_{cyc} h_a^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{2^m.\left(\left(\sum_{cyc} r_a^4\right)\left(\sum_{cyc} r_a^3\right)\right)^m} \overset{\text{Holder}}{\geq} \frac{3^{m+1}.\left(\left(\frac{1}{3^{t-1}}\left(\sum_{cyc} h_a\right)^t\right)^{\frac{n}{m+1}}\right)^{m+1}}{2^m.\left(\left(\sum_{cyc} r_a^4\right)\left(\sum_{cyc} r_a^3\right)\right)^m}$$

$$= \frac{3^{m+1}.\left(\frac{1}{3^{t-1}}\left(\sum_{cyc}\frac{2rs}{a}\right)^t\right)^n}{2^m.\left(\left(\sum_{cyc} r_a^4\right)\left(\sum_{cyc} r_a^3\right)\right)^m} \overset{\text{Bergstrom}}{\geq} \frac{3^{m+1}.\left(\frac{1}{3^{t-1}}\left(2rs.\frac{9}{2s}\right)^t\right)^n}{2^m.\left(\left(\sum_{cyc} r_a^4\right)\left(\sum_{cyc} r_a^3\right)\right)^m}$$

$$=\frac{3^{m+1}\cdot 3^{2nt}\cdot r^{nt}}{3^{nt-n}\cdot 2^m\cdot\left(\left(\sum_{cyc} r_a^4\right)\left(\sum_{cyc} r_a^3\right)\right)^m}\overset{\text{via (2) and (3)}}{\geq}\frac{3^{m+1+2nt-nt+n}\cdot r^{nt}}{2^m\cdot\left(\frac{243}{16}(27R^4-416r^4)\cdot\frac{81}{8}(9R^3-64r^3)\right)^m}$$

$$=\frac{3^{m+1+nt+n-9m}\cdot 2^{7m-m}\cdot r^{nt}}{\left((27R^4-416r^4)(9R^3-64r^3)\right)^m}\therefore\frac{\left(h_a^t+w_b^t+m_c^t\right)^n}{\left(r_a^7+2r_a^3r_b^3(r_a+r_b)+r_b^7\right)^m}$$

$$+\frac{\left(h_b^t+w_c^t+m_a^t\right)^n}{\left(r_b^7+2r_b^3r_c^3(r_b+r_c)+r_c^7\right)^m}+\frac{\left(h_c^t+w_a^t+m_b^t\right)^n}{\left(r_c^7+2r_c^3r_a^3(r_c+r_a)+r_a^7\right)^m}$$

$$\geq\frac{2^{6m}\cdot 3^{n(t+1)-8m+1}\cdot r^{nt}}{\left((27R^4-416r^4)(9R^3-64r^3)\right)^m}\ \forall\,\Delta\,ABC\text{ and }\forall\, m,n,t\in\mathbb{N},$$

$$''=''\ \text{iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1544. In any $\Delta\,ABC$, the following relationship holds :

$$\frac{24r^2}{R^2}\leq\frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}\leq\frac{5106}{256}\left(\frac{R}{r}\right)^6-\frac{5082}{4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}$$

$$\leq\frac{(m_a+m_b)^2}{h_b^2+h_c^2}+\frac{(m_b+m_c)^2}{h_c^2+h_a^2}+\frac{(m_c+m_a)^2}{h_a^2+h_b^2}$$

$$\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\left(\frac{(m_a+m_b)^2+(m_c+m_a)^2}{h_b^2}\right)+\frac{1}{4}\left(\frac{(m_a+m_b)^2+(m_b+m_c)^2}{h_c^2}\right)$$

$$+\frac{1}{4}\left(\frac{(m_b+m_c)^2+(m_c+m_a)^2}{h_a^2}\right)$$

$$\leq\frac{2}{4}\left(\frac{m_a^2+m_b^2+m_c^2+m_a^2}{h_b^2}\right)+\frac{2}{4}\left(\frac{m_a^2+m_b^2+m_b^2+m_c^2}{h_c^2}\right)$$

$$+\frac{2}{4}\left(\frac{m_b^2+m_c^2+m_c^2+m_a^2}{h_a^2}\right)$$

$$=\frac{1}{2}\cdot\left(\sum_{cyc} m_a^2\right)\cdot\frac{1}{4r^2s^2}\cdot\left(\sum_{cyc} a^2\right)$$

$$+\frac{1}{2}\cdot\frac{1}{16r^2s^2}\left(a^2(2a^2+2b^2-c^2)+b^2(2b^2+2c^2-a^2)+c^2(2c^2+2a^2-b^2)\right)$$

$$\overset{\text{Leibnitz}}{\leq}\frac{1}{2}\cdot\frac{3}{4}\cdot 9R^2\cdot\frac{1}{4r^2s^2}\cdot 9R^2+\frac{1}{32r^2s^2}\left(5\sum_{cyc} a^2b^2-32r^2s^2\right)\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Goldstone}}}{\leq}$$

$$\frac{243R^4}{32r^2.27r^2}+\frac{5.4R^2s^2-32r^2s^2}{32r^2s^2}=\frac{9R^4+20R^2r^2-32r^4}{32r^4}\overset{?}{\le}\frac{5106R^6-64.5082r^6}{256r^6}$$

$$\Leftrightarrow 2553t^6-36t^4-80t^2-162496\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2553t^5+5106t^4+10176t^3+20352t^2+40624t+81248)\overset{?}{\ge}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\ge}2\therefore\frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}\le\frac{5106}{256}\left(\frac{R}{r}\right)^6-\frac{5082}{4}$$

$$\text{Again,}\frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}$$

$$\ge\frac{(h_a+h_b)^2}{m_b^2+m_c^2}+\frac{(h_b+h_c)^2}{m_c^2+m_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}\overset{\text{Bergstrom}}{\ge}\frac{4\left(\sum_{cyc}\frac{2rs}{a}\right)^2}{2\sum_{cyc}m_a^2}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Leibnitz}}}{\ge}\frac{4\left(\frac{2rs.9}{2s}\right)^2}{2.\frac{3}{4}.9R^2}$$

$$=\frac{24r^2}{R^2}\therefore\frac{24r^2}{R^2}\le\frac{(m_a+m_b)^2}{w_b^2+w_c^2}+\frac{(w_b+w_c)^2}{h_c^2+h_a^2}+\frac{(h_c+h_a)^2}{m_a^2+m_b^2}$$

$$\le\frac{5106}{256}\left(\frac{R}{r}\right)^6-\frac{5082}{4}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1545. In any Δ ABC, the following relationship holds :

$$\frac{144r^3}{R^2}\le\frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}\le\frac{9(243R^7-30976r^7)}{32r^6}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}$$

$$\le\frac{(m_a+m_b)^3}{h_b^2+h_c^2}+\frac{(m_b+m_c)^3}{h_c^2+h_a^2}+\frac{(m_c+m_a)^3}{h_a^2+h_b^2}$$

$$\overset{\text{Reverse Bergstrom}}{\le}\frac{1}{4}\left(\frac{(m_a+m_b)^3+(m_c+m_a)^3}{h_b^2}+\frac{(m_a+m_b)^3+(m_b+m_c)^3}{h_c^2}+\frac{(m_b+m_c)^3+(m_c+m_a)^3}{h_a^2}\right)$$

$$\overset{\text{Holder}}{\le}\frac{4}{4}\left(\frac{m_a^3+m_b^3+m_c^3+m_a^3}{h_b^2}+\frac{m_a^3+m_b^3+m_b^3+m_c^3}{h_c^2}+\frac{m_b^3+m_c^3+m_c^3+m_a^3}{h_a^2}\right)$$

$$=\frac{2\sum_{cyc}m_a^3-(m_b^3+m_c^3)}{h_b^2}+\frac{2\sum_{cyc}m_a^3-(m_c^3+m_a^3)}{h_c^2}+\frac{2\sum_{cyc}m_a^3-(m_a^3+m_b^3)}{h_a^2}$$

$$=2\left(\sum_{cyc}m_a^3\right)\left(\frac{\sum_{cyc}a^2}{4r^2s^2}\right)-\left(\frac{m_b^3}{h_b^2}+\frac{m_c^3}{h_b^2}+\frac{m_c^3}{h_c^2}+\frac{m_a^3}{h_c^2}+\frac{m_a^3}{h_a^2}+\frac{m_b^3}{h_a^2}\right)$$

$$\overset{\substack{\text{Radon}\\ \text{and}\\ \text{Leibnitz}}}{\leq}2\left((4R+r)^3-3(m_a+m_b)(m_b+m_c)(m_c+m_a)\right)\left(\frac{9R^2}{4r^2s^2}\right)-\frac{\left(2\sum_{cyc}m_a\right)^3}{\left(2\sum_{cyc}h_a\right)^2}$$

$$\overset{\substack{\text{Euler,}\\ \text{Mitrinovic}\\ \text{and}\\ \text{Cesaro}}}{\leq}2\left(\left(\frac{9R}{2}\right)^3-24h_ah_bh_c\right)\left(\frac{9R^2}{4r^2.27r^2}\right)-2\sum_{cyc}\frac{2rs}{a}$$

$$\overset{\text{Bergstrom}}{\leq}2\left(\left(\frac{9R}{2}\right)^3-24.\frac{2r^2s^2}{R}\right)\left(\frac{9R^2}{4r^2.27r^2}\right)-2.\frac{2rs.9}{2s}$$

$$\overset{\text{Gerretsen + Euler}}{\leq}2\left(\left(\frac{9R}{2}\right)^3-24.\frac{r^2.27Rr}{R}\right)\left(\frac{9R^2}{4r^2.27r^2}\right)-18r$$

$$=9\left(\frac{3R^2(9R^3-64r^3)-32r^5}{16r^4}\right)\overset{?}{\leq}\frac{9(243R^7-30976r^7)}{32r^6}$$

$$\Leftrightarrow 243t^7-54t^5+384t^2-30912\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(243t^6+486t^5+918t^4+1836t^3+3672t^2+7728t+15456)\overset{?}{\geq}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\geq}2$$

$$\therefore \frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}\leq\frac{9(243R^7-30976r^7)}{32r^6}$$

$$\text{Again, }\frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}$$

$$\geq\frac{(h_a+h_b)^3}{m_b^2+m_c^2}+\frac{(h_b+h_c)^3}{m_c^2+m_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}\overset{\text{Holder}}{\geq}\frac{8\left(\sum_{cyc}\frac{2rs}{a}\right)^3}{3.2\sum_{cyc}m_a^2}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Leibnitz}}}{\geq}\frac{8\left(\frac{2rs.9}{2s}\right)^3}{3.2.\frac{3}{4}.9R^2}$$

$$=\frac{144r^3}{R^2}\therefore\frac{144r^3}{R^2}\leq\frac{(m_a+m_b)^3}{w_b^2+w_c^2}+\frac{(w_b+w_c)^3}{h_c^2+h_a^2}+\frac{(h_c+h_a)^3}{m_a^2+m_b^2}$$

$$\leq\frac{9(243R^7-30976r^7)}{32r^6}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1546.

In ΔABC the following relationship holds:

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}}\geq\frac{r_a}{r_b}+\frac{r_b}{r_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}}\right)-\left(\frac{r_a}{r_b}+\frac{r_b}{r_a}\right)=\frac{a(s-b)}{b(s-a)}+\frac{b(s-a)}{a(s-b)}-\frac{s-b}{s-a}-\frac{s-a}{s-b}$$

$$=\frac{s-b}{s-a}\left(\frac{a}{b}-1\right)-\frac{s-a}{s-b}\left(1-\frac{b}{a}\right)=\frac{(a-b)(s-b)}{b(s-a)}-\frac{(a-b)(s-a)}{a(s-b)}$$

$$=\frac{(a-b)[a(s-b)^2-b(s-a)^2]}{ab(s-a)(s-b)}=\frac{(a-b)^2(s^2-ab)}{ab(s-a)(s-b)}\overset{s>a,b}{\geq}0,$$

Therefore

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}}\geq\frac{r_a}{r_b}+\frac{r_b}{r_a}.$$

Equality holds iff $a=b$.

1547. In any Δ ABC, the following relationship holds :

$$\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\geq\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)=\frac{3F}{2a}+\frac{F}{b}-\frac{Rp}{4c}\overset{\text{Panaitopol}}{\leq}\frac{3}{4}h_a+\frac{1}{2}h_b-\frac{m_c}{4}$$

$$\leq\frac{3}{4}m_a+\frac{1}{2}m_b-\frac{m_c}{4}\therefore\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)\leq\frac{3m_a+2m_b-m_c}{4}\to(1)$$

$$\text{Again,}\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\overset{\text{Bergstrom}}{\geq}\frac{(m_a+m_b)^2}{m_a+2m_b+m_c}\overset{?}{\geq}\frac{3m_a+2m_b-m_c}{4}$$

$$\Leftrightarrow 4(x+y)^2\overset{?}{\geq}(3x+2y-z)(x+2y+z)\ (x=m_a,y=m_b,z=m_c)$$

$$\Leftrightarrow 4x^2+4y^2+8xy\overset{?}{\geq}3x^2+6xy+3xz+2xy+4y^2+2yz-zx-2yz-z^2$$

$$\Leftrightarrow x^2-2xz+z^2\overset{?}{\geq}0\Leftrightarrow(x-z)^2\overset{?}{\geq}0\to\text{true}$$

$$\therefore\frac{m_a^2}{m_a+m_b}+\frac{m_b^2}{m_b+m_c}\geq\frac{3m_a+2m_b-m_c}{4}\overset{\text{via }(1)}{\geq}\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right)$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{{m_a}^2}{m_a+m_b}+\frac{m_a+m_b}{4}\geq m_a \text{ and } \frac{{m_b}^2}{m_b+m_c}+\frac{m_b+m_c}{4}\geq m_b.$$

Adding these inequalities, we have

$$\frac{{m_a}^2}{m_a+m_b}+\frac{{m_b}^2}{m_b+m_c}\geq\frac{1}{4}(3m_a+2m_b-m_c).$$

Also, we have $m_a\geq h_a=\frac{2pr}{a},\ \ m_b\geq h_b=\frac{2pr}{b}$ and $m_c\leq\frac{Rh_c}{2r}=\frac{pR}{c}$.

Therefore

$$\frac{{m_a}^2}{m_a+m_b}+\frac{{m_b}^2}{m_b+m_c}\geq\frac{p}{4}\left(\frac{6r}{a}+\frac{4r}{b}-\frac{R}{c}\right).$$

Equality holds iff ΔABC is equilateral.

1548. In any Δ ABC, the following relationship holds :

$$\frac{576r^4}{9R^3-64r^3}\leq\frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}\leq\frac{9(19683R^{10}-20154368r^{10})}{128r^9}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\leq\frac{(m_a+m_b)^4}{h_b^3+h_c^3}+\frac{(m_b+m_c)^4}{h_c^3+h_a^3}+\frac{(m_c+m_a)^4}{h_a^3+h_b^3}$$

$$\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\left(\frac{(m_a+m_b)^4+(m_c+m_a)^4}{h_b^3}\right)+\frac{1}{4}\left(\frac{(m_a+m_b)^4+(m_b+m_c)^4}{h_c^3}\right)$$

$$+\frac{1}{4}\left(\frac{(m_b+m_c)^4+(m_c+m_a)^4}{h_a^3}\right)$$

$$\overset{\text{Holder}}{\leq}\frac{8}{4}\left(\frac{m_a^4+m_b^4+m_c^4+m_a^4}{h_b^3}+\frac{m_a^4+m_b^4+m_b^4+m_c^4}{h_c^3}+\frac{m_b^4+m_c^4+m_c^4+m_a^4}{h_a^3}\right)$$

$$=2\left(\frac{2\sum_{cyc}m_a^4-(m_b^4+m_c^4)}{h_b^3}+\frac{2\sum_{cyc}m_a^4-(m_c^4+m_a^4)}{h_c^3}+\frac{2\sum_{cyc}m_a^4-(m_a^4+m_b^4)}{h_a^3}\right)$$

$$=4\left(\sum_{cyc}m_a^4\right)\left(\frac{\sum_{cyc}a^3}{8r^3s^3}\right)-2\left(\frac{m_b^4}{h_b^3}+\frac{m_c^4}{h_b^3}+\frac{m_c^4}{h_c^3}+\frac{m_a^4}{h_c^3}+\frac{m_a^4}{h_a^3}+\frac{m_b^4}{h_a^3}\right)$$

$$\overset{\text{Radon}}{\leq}4.\frac{9}{16}.\left(\left(\sum_{cyc}a^2\right)^2-2\sum_{cyc}a^2b^2\right).\frac{2s(s^2-6Rr-3r^2)}{8r^3s^3}-2.\frac{\left(2\sum_{cyc}m_a\right)^4}{\left(2\sum_{cyc}h_a\right)^3}$$

$$\overset{\substack{\text{Leibnitz,}\\ \text{Gordon,}\\ \text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen}}}{\leq} 4.\frac{9}{16}.\left(81R^4-\frac{2}{3}.48r^2.27r^2\right).\frac{4R^2-2Rr}{4r^3.27r^2}-4.\sum_{cyc}\frac{2rs}{a}\overset{\text{Bergstrom}}{\leq}$$

$$\frac{9(81R^4-864r^4)(2R^2-Rr)}{8r^3.27r^2}-4.\frac{2rs.9}{2s}=9.\frac{(3R^4-32r^4)(2R^2-Rr)-32r^6}{8r^5}$$

$$\overset{?}{\leq}\frac{9(19683R^{10}-20154368r^{10})}{128r^9}$$

$$\Leftrightarrow 19683t^{10}-96t^6+48t^5+1024t^2-512t-20153856\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow$$

$$(t-2)\left(\begin{matrix}19683t^9+39366t^8+78732t^7+157464t^6+314832t^5\\ +629712t^4+1259424t^3+2518848t^2+5038720t+10076928\end{matrix}\right)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\ \therefore \frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\leq\frac{9(19683R^{10}-20154368r^{10})}{128r^9}$$

$$\text{Again},\frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\geq\frac{(h_a+h_b)^4}{m_b^3+m_c^3}+\frac{(h_b+h_c)^4}{m_c^3+m_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\overset{\text{Holder}}{\geq}\frac{16\left(\sum_{cyc}\frac{2rs}{a}\right)^4}{9.2\sum_{cyc}m_a^3}\overset{\substack{\text{Bergstrom,}\\ \text{Leuenberger}\\ \text{and}\\ \text{Cesaro}}}{\geq}\frac{8\left(\frac{2rs.9}{2s}\right)^4}{9((4R+r)^3-24m_am_bm_c)}\overset{\text{Euler}}{\geq}\frac{8.729r^4}{\left(\frac{9R}{2}\right)^3-24h_ah_bh_c}$$

$$=\frac{8.729r^4}{\left(\frac{9R}{2}\right)^3-24.\frac{2r^2s^2}{R}}\overset{\text{Gerretsen + Euler}}{\geq}\frac{8.729r^4}{\left(\frac{9R}{2}\right)^3-24.\frac{r^2.27Rr}{R}}=\frac{576r^4}{9R^3-64r^3}$$

$$\therefore\frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}\geq\frac{576r^4}{9R^3-64r^3}$$

$$\therefore\frac{576r^4}{9R^3-64r^3}\leq\frac{(m_a+m_b)^4}{w_b^3+w_c^3}+\frac{(w_b+w_c)^4}{h_c^3+h_a^3}+\frac{(h_c+h_a)^4}{m_a^3+m_b^3}$$

$$\leq\frac{9(19683R^{10}-20154368r^{10})}{128r^9}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1549.

In any Δ ABC, the following relationship holds :

$$5+\sum_{cyc}\frac{n_bn_c}{r_br_c}\leq\frac{4R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$

$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_ar_a \text{ (and analogs) and } \frac{1}{r_b}+\frac{1}{r_c}=\frac{2}{h_a} \text{ (and analogs).}$$

Using these identities, we have

$$\sum_{cyc}\frac{n_bn_c}{r_br_c}\overset{AM-GM}{\leq}\sum_{cyc}\frac{n_b{}^2+n_c{}^2}{2r_br_c}=\sum_{cyc}\frac{(s^2-2h_br_b)+(s^2-2h_cr_c)}{2r_br_c}=\sum_{cyc}\left(\frac{s^2}{r_br_c}-\left(\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\right)$$

$$=\frac{s^2(r_a+r_b+r_c)}{r_ar_br_c}-\sum_{cyc}\left(\frac{h_a}{r_b}+\frac{h_a}{r_c}\right)=\frac{s^2(4R+r)}{s^2r}-\sum_{cyc}2=\frac{4R}{r}-5,$$

which completes the proof. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_ar_a\Rightarrow h_ar_a=\frac{s^2-n_a^2}{2}\text{ and analogs}\Rightarrow n_bn_c+h_br_b+h_cr_c$$
$$=n_bn_c+\frac{s^2-n_b^2}{2}+\frac{s^2-n_c^2}{2}=\frac{2s^2-(n_b^2+n_c^2-n_bn_c)}{2}=\frac{2s^2-(n_b-n_c)^2}{2}\leq\frac{2s^2}{2}$$

$$\Rightarrow\frac{n_bn_c+h_br_b+h_cr_c}{r_br_c}\leq\frac{s^2}{s(s-a)}\Rightarrow\frac{n_bn_c}{r_br_c}\overset{(*)}{\leq}\frac{s}{s-a}-\frac{h_b}{r_c}-\frac{h_c}{r_b}\text{ and analogously,}$$

$$\frac{n_c n_a}{r_c r_a} \overset{(**)}{\leq} \frac{s}{s-b} - \frac{h_c}{r_a} - \frac{h_a}{r_c} \text{ and } \frac{n_a n_b}{r_a r_b} \overset{(***)}{\leq} \frac{s}{s-c} - \frac{h_a}{r_b} - \frac{h_b}{r_a} \therefore (*) + (**) + (***) \Rightarrow$$

$$\sum_{cyc} \frac{n_b n_c}{r_b r_c} \leq \sum_{cyc} \frac{s}{s-a} - \sum_{cyc} \frac{h_a(r_b + r_c)}{r_b r_c} \overset{\text{via (i) and analogs}}{=} \frac{s}{r^2 s} \cdot \sum_{cyc} (s-b)(s-c)$$

$$-\sum_{cyc} \frac{\frac{bc}{2R} \cdot 4R\cos^2\frac{A}{2}}{bc.\cos^2\frac{A}{2}} = \frac{4Rr + r^2}{r^2} - 6 \Rightarrow 5 + \sum_{cyc} \frac{n_b n_c}{r_b r_c} \leq \frac{4R}{r} + 1 - 6 + 5 = \frac{4R}{r}$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1550. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \sqrt{\frac{1}{2}\left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1\right)} \leq \sqrt{\frac{2R - r + AI + BI + CI}{r}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $n_a{}^2 = s(s-a) + \frac{s(b-c)^2}{a} = s^2 - \frac{s[a^2 - (b-c)^2]}{a} = s^2 - \frac{4s(s-b)(s-c)}{a}$

$$= s^2 - \frac{4s.sr^2}{a(s-a)} = s^2 - 2h_a r_a \text{ (and analogs)},$$

then

$$\frac{n_b n_c}{r_b r_c} \overset{AM-GM}{\leq} \frac{n_b{}^2 + n_c{}^2}{2r_b r_c} = \frac{(s^2 - 2h_b r_b) + (s^2 - 2h_c r_c)}{2r_b r_c} = \frac{s^2}{r_b r_c} - \left(\frac{h_b}{r_c} + \frac{h_c}{r_b}\right) \text{ (and analogs)}.$$

Using this inequality, we have

$$\sum_{cyc} \sqrt{\frac{1}{2}\left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1\right)} \leq \sum_{cyc} \sqrt{\frac{1}{2}\left(\frac{s^2}{r_b r_c} - 1\right)} = \sum_{cyc} \sqrt{\frac{1}{2}\left(\frac{s}{s-a} - 1\right)} = \sum_{cyc} \sqrt{\frac{a}{2(s-a)}}$$

$$= \sqrt{\sum_{cyc} \frac{a}{2(s-a)} + 2\sum_{cyc} \sqrt{\frac{bc}{2(s-b).2(s-c)}}} = \sqrt{\frac{2R-r}{r} + \sum_{cyc} \frac{1}{\sin\frac{A}{2}}} = \sqrt{\frac{2R - r + AI + BI + CI}{r}},$$

as desired. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$:\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_ar_a\Rightarrow h_ar_a=\frac{s^2-n_a^2}{2}\text{ and analogs}\Rightarrow n_bn_c+h_br_b+h_cr_c$$
$$=n_bn_c+\frac{s^2-n_b^2}{2}+\frac{s^2-n_c^2}{2}=\frac{2s^2-(n_b^2+n_c^2-n_bn_c)}{2}=\frac{2s^2-(n_b-n_c)^2}{2}\le\frac{2s^2}{2}$$
$$\Rightarrow\frac{n_bn_c+h_br_b+h_cr_c}{r_br_c}\le\frac{s^2}{s(s-a)}$$

$$\Rightarrow\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sqrt{\frac{1}{2}\left(\frac{s^2}{s(s-a)}-1\right)}=\sqrt{\frac{1}{2}\cdot\frac{a}{s-a}}$$

$$=\sqrt{\frac{1}{2}\cdot\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}=\sqrt{\frac{1}{2}\cdot\frac{\sin^2\frac{A}{2}}{\frac{r}{4R}}}$$
$$\Rightarrow\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sqrt{\frac{2R}{r}}.\sin\frac{A}{2}\text{ and analogs}$$
$$\Rightarrow\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}\le\sqrt{\frac{2R}{r}}.\sum_{cyc}\sin\frac{A}{2}\overset{?}{\le}\sqrt{\frac{2R-r+AI+BI+CI}{r}}$$
$$\Leftrightarrow\sum_{cyc}\sin\frac{A}{2}\overset{?}{\le}\sqrt{\frac{2R-r+AI+BI+CI}{2R}}=\sqrt{\frac{2R-r}{2R}+\sum_{cyc}\left(\frac{r}{2R}\cdot\frac{1}{\sin\frac{A}{2}}\right)}$$
$$=\sqrt{\sum_{cyc}\sin^2\frac{A}{2}+\sum_{cyc}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cdot\frac{1}{\sin\frac{A}{2}}\right)}=\sqrt{\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}}$$
$$\Leftrightarrow\sum_{cyc}\sin\frac{A}{2}\overset{?}{\le}\sqrt{\left(\sum_{cyc}\sin\frac{A}{2}\right)^2}\rightarrow\text{true}\therefore\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}-1\right)}$$

$$\leq\sqrt{\frac{2R-r+AI+BI+CI}{r}}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1551. In any Δ ABC, the following relationship holds :

$$\prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\leq\frac{s^2}{r^2}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $n_a{}^2=s(s-a)+\frac{s(b-c)^2}{a}=s^2-\frac{s[a^2-(b-c)^2]}{a}=s^2-\frac{4s(s-b)(s-c)}{a}$

$$=s^2-\frac{4s.sr^2}{a(s-a)}=s^2-2h_a r_a \text{ (and analogs)},$$

then

$$\frac{n_b n_c}{r_b r_c}\overset{AM-GM}{\leq}\frac{n_b{}^2+n_c{}^2}{2r_b r_c}=\frac{(s^2-2h_b r_b)+(s^2-2h_c r_c)}{2r_b r_c}=\frac{s^2}{r_b r_c}-\left(\frac{h_b}{r_c}+\frac{h_c}{r_b}\right) \text{ (and analogs)}.$$

Using this inequality, we have

$$\prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\leq\prod_{cyc}\left(\frac{s^2}{r_b r_c}\right)=\frac{(s^2)^3}{(s^2 r)^2}=\frac{s^2}{r^2},$$

as desired. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$
$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$
$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$
$$=as^2-2ah_a r_a\Rightarrow h_a r_a=\frac{s^2-n_a^2}{2} \text{ and analogs}\Rightarrow n_b n_c+h_b r_b+h_c r_c$$
$$=n_b n_c+\frac{s^2-n_b^2}{2}+\frac{s^2-n_c^2}{2}=\frac{2s^2-(n_b^2+n_c^2-n_b n_c)}{2}=\frac{2s^2-(n_b-n_c)^2}{2}\leq\frac{2s^2}{2}$$
$$\Rightarrow\frac{n_b n_c+h_b r_b+h_c r_c}{r_b r_c}\leq\frac{s^2}{s(s-a)}\Rightarrow\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\leq\frac{s}{s-a} \text{ and analogs}$$
$$\Rightarrow\prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\leq\frac{s^3}{(s-a)(s-b)(s-c)}=\frac{s^3}{r^2 s}$$

$$\therefore \prod_{cyc}\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\le\frac{s^2}{r^2}\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

1552. In any Δ ABC, the following relationship holds :

$$9r\le\sum_{cyc}\frac{m_b^3+m_c^3}{w_b^2+w_c^2}\le\frac{9R^5}{32r^4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG if we assume $a\ge b\ge c$, then : $m_b^3+m_c^3\ge m_c^3+m_a^3\ge m_a^3+m_b^3$

$$\text{and } \frac{1}{w_b^2+w_c^2}\le\frac{1}{w_c^2+w_a^2}\le\frac{1}{w_a^2+w_b^2}\ \therefore\ \sum_{cyc}\frac{m_b^3+m_c^3}{w_b^2+w_c^2}\overset{\text{Chebyshev}}{\le}$$

$$\frac{1}{3}\left(\sum_{cyc}(m_b^3+m_c^3)\right)\left(\sum_{cyc}\frac{1}{w_b^2+w_c^2}\right)\overset{A-G}{\le}$$

$$\frac{2}{3}\left(m_a\left(\frac{2\sum_{cyc}a^2-3a^2}{4}\right)+m_b\left(\frac{2\sum_{cyc}a^2-3b^2}{4}\right)+m_c\left(\frac{2\sum_{cyc}a^2-3c^2}{4}\right)\right)\left(\frac{1}{2}\sum_{cyc}\frac{1}{w_b w_c}\right)$$

$$\le\frac{1}{3}\left(\left(\frac{\sum_{cyc}a^2}{2}\right)\left(\sum_{cyc}m_a\right)-\frac{3}{4}\sum_{cyc}\frac{a^2}{\frac{1}{m_a}}\right)\left(\sum_{cyc}\frac{1}{w_a^2}\right)\overset{\substack{\text{Leuenberger + Euler}\\ \text{and}\\ \text{Bergstrom}}}{\le}$$

$$\frac{1}{3}\left((s^2-4Rr-r^2).\frac{9R}{2}-\frac{3}{4}.\frac{4s^2}{\sum_{cyc}\frac{1}{h_a}}\right)\left(\sum_{cyc}\frac{1}{h_a^2}\right)$$

$$=\frac{1}{3}\left((s^2-4Rr-r^2).\frac{9R}{2}-3rs^2\right)\left(\frac{\sum_{cyc}a^2}{4r^2s^2}\right)$$

$$\overset{\text{Leibnitz}}{\le}\frac{(3R-2r)s^2-3R(4Rr+r^2)}{2}.\frac{9R^2}{4r^2s^2}\overset{?}{\le}\frac{9R^5}{32r^4}$$

$$\Leftrightarrow (R^3-12Rr^2+8r^3)s^2+r^3(48R^2+12Rr)\underset{(*)}{\overset{?}{\ge}}0$$

$\boxed{Case\ 1}$ $R^3-12Rr^2+8r^3\ge0$ and then : LHS of $(*)\ge r^3(48R^2+12Rr)>0$
$\Rightarrow(*)$ is true (strict inequality)

$\boxed{Case\ 2}$ $R^3-12Rr^2+8r^3<0$ and then : LHS of $(*)$

$$=-\left(-(R^3-12Rr^2+8r^3)\right)s^2+r^3(48R^2+12Rr)\overset{\text{Gerretsen}}{\ge}$$

$$-\left(-\left(R^3-12Rr^2+8r^3\right)\right)\left(4R^2+4Rr+3r^2\right)+r^3\left(48R^2+12Rr\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 4t^5+4t^4-45t^3+32t^2+8t+24\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\left(4t^3+20t^2+19t+28\right)+44\right)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)$$

is true and combining both cases, $(*)\ \forall\ \Delta\ ABC\ \therefore \sum_{cyc}\frac{m_b^3+m_c^3}{w_b^2+w_c^2}\leq\frac{9R^5}{32r^4}\ \forall\ \Delta\ ABC$

$$\text{Again,}\sum_{cyc}\frac{m_b^3+m_c^3}{w_b^2+w_c^2}\geq\sum_{cyc}\frac{w_b^3+w_c^3}{w_b^2+w_c^2}\overset{\text{Chebyshev}}{\geq}\sum_{cyc}\frac{\frac{1}{2}\left(w_b^2+w_c^2\right)\left(w_b+w_c\right)}{w_b^2+w_c^2}=\sum_{cyc}w_a$$

$$\geq\sum_{cyc}h_a=\sum_{cyc}\frac{2rs}{a}\overset{\text{Bergstrom}}{\geq}\frac{2rs.9}{2s}\ \therefore\sum_{cyc}\frac{m_b^3+m_c^3}{w_b^2+w_c^2}\geq 9r$$

$$\therefore 9r\leq\sum_{cyc}\frac{m_b^3+m_c^3}{w_b^2+w_c^2}\leq\frac{9R^5}{32r^4}\ \forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\text{ is equilateral (QED)}$$

1553. In ΔABC holds :

$$9r\leq\sum_{cyc}\frac{{m_b}^3+{m_c}^3}{{w_b}^2+{w_c}^2}\leq\frac{9R^5}{32r^4}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{{m_b}^3+{m_c}^3}{{w_b}^2+{w_c}^2}\overset{m_a\geq w_a}{\geq}\sum_{cyc}\frac{{w_b}^3+{w_c}^3}{{w_b}^2+{w_c}^2}\overset{\text{Chebyshev}}{\geq}\sum_{cyc}\frac{w_b+w_c}{2}\overset{w_a\geq h_a}{\geq}\sum_{cyc}h_a\overset{AM-HM}{\geq}\frac{9}{\sum_{cyc}\frac{1}{h_a}}=9r.$$

$$\sum_{cyc}\frac{{m_b}^3+{m_c}^3}{{w_b}^2+{w_c}^2}\overset{\substack{\text{Panaitopol}\\ AM-GM}}{\leq}\sum_{cyc}\frac{\frac{Rh_b}{2r}.{m_b}^2+\frac{Rh_c}{2r}.{m_c}^2}{2w_bw_c}\overset{w_a\leq h_a}{\leq}\frac{R}{2r}\sum_{cyc}\frac{h_b{m_b}^2+h_c{m_c}^2}{2h_bh_c}$$

$$=\frac{R}{4r}\sum_{cyc}\left(\frac{{m_b}^2}{h_c}+\frac{{m_c}^2}{h_b}\right)=\frac{R}{4r}\sum_{cyc}\left(\frac{{m_a}^2}{h_b}+\frac{{m_a}^2}{h_c}\right)=\frac{R}{4r}\sum_{cyc}\frac{b+c}{2F}.\frac{2b^2+2c^2-a^2}{4}$$

$$=\frac{R}{32sr^2}\left(3\sum_{cyc}a^3+\sum_{cyc}a.\sum_{cyc}a^2\right)=\frac{R}{32sr^2}\left[3.2s\left(s^2-3r^2-6Rr\right)+2s.2\left(s^2-r^2-4Rr\right)\right]$$

$$=\frac{R\left(5s^2-11r^2-26Rr\right)}{16r^2}\overset{\text{Gerretsen}}{\leq}\frac{R\left(20R^2-4r^2-6Rr\right)}{16r^2}\overset{?}{\leq}\frac{9R^5}{32r^4}$$

$$\Leftrightarrow R(R-2r)\left(9R^3+18R^2r-4Rr^2+4r^3\right)\geq 0,$$

which is true by Euler's inequality.

So the proof is complete. Equality holds iff ΔABC is equilateral.

1554. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\tan^4\frac{A}{2}+\tan^4\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\ge 1$$

Proposed by Marin Chirciu-Romania

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{\tan^4\frac{A}{2}+\tan^4\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}=\sum_{cyc}\frac{\tan^4\frac{A}{2}+\tan^4\frac{B}{2}}{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}\cdot\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\ge$$

$$\overset{LEHMER}{\ge}\sum_{cyc}\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\cdot\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}=$$

$$=\sum_{cyc}\left(\frac{\tan^3\frac{A}{2}+\tan^3\frac{B}{2}}{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}\right)^2\overset{LEHMER}{\ge}\sum_{cyc}\left(\frac{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}\right)^2\ge$$

$$\overset{LEHMER}{\ge}\sum_{cyc}\left(\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1+1}\right)^2\overset{AM-GM}{\ge}\sum_{cyc}\left(\sqrt{\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\right)^2=$$

$$=\sum_{cyc}\tan\frac{A}{2}\cdot\tan\frac{B}{2}=1$$

Equality holds for $A=B=C=\frac{\pi}{3}$.

1555. In ΔABC the following relationship holds:

$$\frac{1}{\mu(A)\cdot\cos^2\frac{A}{2}}+\frac{1}{\mu(B)\cdot\cos^2\frac{B}{2}}+\frac{1}{\mu(C)\cdot\cos^2\frac{C}{2}}\ge\frac{12}{\pi}$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Daniel Sitaru-Romania

$$\frac{1}{\mu(A)\cdot\cos^2\frac{A}{2}}+\frac{1}{\mu(B)\cdot\cos^2\frac{B}{2}}+\frac{1}{\mu(C)\cdot\cos^2\frac{C}{2}}=$$

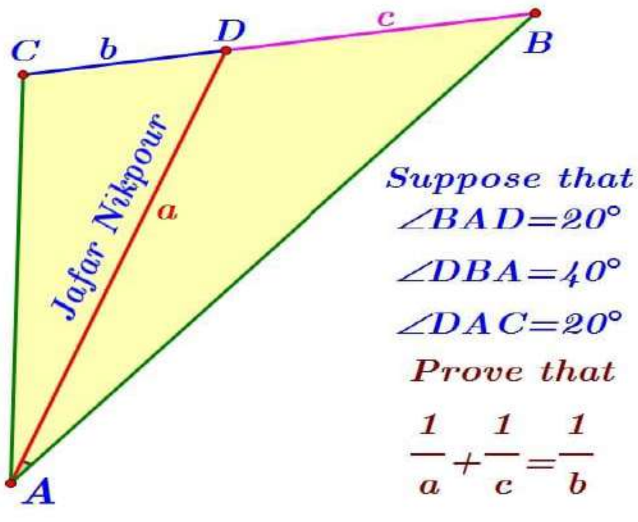
$$=\frac{\frac{1}{\cos^2\frac{A}{2}}}{\mu(A)}+\frac{\frac{1}{\cos^2\frac{B}{2}}}{\mu(B)}+\frac{\frac{1}{\cos^2\frac{C}{2}}}{\mu(C)}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{\left(\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\right)^2}{\mu(A)+\mu(B)+\mu(C)}=\frac{1}{\pi}\left(\frac{1}{\cos\frac{A}{2}}+\frac{1}{\cos\frac{B}{2}}+\frac{1}{\cos\frac{C}{2}}\right)^2\geq$$

$$\overset{JENSEN}{\geq}\frac{1}{\pi}\left(3\cdot\frac{1}{\cos\left(\frac{A+B+C}{6}\right)}\right)^2=\frac{1}{\pi}\left(3\cdot\frac{1}{\cos\left(\frac{\pi}{6}\right)}\right)^2=\frac{1}{\pi}\left(3\cdot\frac{1}{\frac{\sqrt{3}}{2}}\right)^2=\frac{12}{\pi}$$

Equality holds for $A=B=C=\frac{\pi}{3}$.

1556.



Proposed by Jafar Nikpour-Iran

Solution by Daniel Sitaru-Romania

$$m(\sphericalangle ABD)=180^\circ-40^\circ-20^\circ=120^\circ,\ m(\sphericalangle ADC)=180^\circ-120^\circ=60^\circ$$

In ΔADC:

$$\frac{a}{\sin 100^\circ}=\frac{b}{\sin 20^\circ}\Rightarrow b=\frac{a\sin 20^\circ}{\sin 100^\circ}$$

In ΔADB:

$$\frac{c}{sin20^\circ}=\frac{a}{sin40^\circ}\Rightarrow c=\frac{asin20^\circ}{sin40^\circ}$$

$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b}\Leftrightarrow\frac{1}{a}+\frac{1}{\frac{asin20^\circ}{sin100^\circ}}=\frac{1}{\frac{asin20^\circ}{sin40^\circ}}\Leftrightarrow sin20^\circ+sin40^\circ=sin100^\circ\Leftrightarrow$$

$$\Leftrightarrow 2sin\frac{20^\circ+40^\circ}{2}cos\frac{20^\circ-40^\circ}{2}=sin(90^\circ+10^\circ)\Leftrightarrow$$

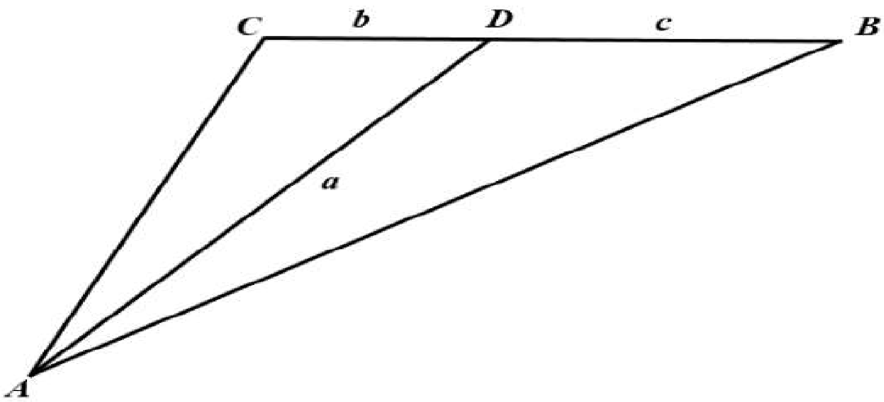
$$\Leftrightarrow 2sin30^\circ cos10^\circ=sin90^\circ cos10^\circ+sin10^\circ cos90^\circ\Leftrightarrow$$

$$\Leftrightarrow 2\cdot\frac{1}{2}cos10^\circ=1\cdot cos10^\circ+sin10^\circ\cdot 0\Leftrightarrow cos10^\circ=cos10^\circ$$

1557. Suppose that ∠DBA=sin(50^0), ∠BAD=sin(10^0), ∠DAC=sin(10^0)

Prove that

$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b}$$



Proposed by Jafar Nikpour-Iran

Solution by Togrul Ehmedov-Azerbaijan

Using the law of sines, we conclude that

$$\begin{cases}\frac{\sin(110^0)}{a}=\frac{\sin(10^0)}{b}\\ \frac{\sin(50^0)}{a}=\frac{\sin(10^0)}{c}\end{cases}\Rightarrow\begin{cases}\frac{1}{a}=\frac{\sin(10^0)}{\sin(50^0)}*\frac{1}{c}\\ \frac{1}{b}=\frac{\sin(110^0)}{\sin(50^0)}*\frac{1}{c}\end{cases}$$

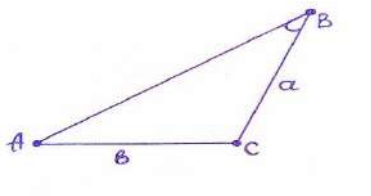
$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b}\Rightarrow\frac{\sin(10^0)}{\sin(50^0)}*\frac{1}{c}+\frac{1}{c}=\frac{\sin(110^0)}{\sin(50^0)}*\frac{1}{c}\Rightarrow\frac{\sin(10^0)}{\sin(50^0)}+1=\frac{\sin(110^0)}{\sin(50^0)}$$

$$\frac{\sin(10^0)+\sin(50^0)}{\sin(50^0)}=\frac{\sin(110^0)}{\sin(50^0)}$$

$$\frac{2\sin\left(\frac{10^0+50^0}{2}\right)\cos\left(\frac{10^0-50^0}{2}\right)}{\sin(50^0)}=\frac{\sin(90^0+20^0)}{\sin(50^0)}$$

$$\frac{2\sin(30^0)\cos(20^0)}{\sin(50^0)}=\frac{\cos(20^0)}{\sin(50^0)}\Rightarrow\frac{\cos(20^0)}{\sin(50^0)}=\frac{\cos(20^0)}{\sin(50^0)}$$

1558. **Suppose that ∠ABC=20^0, ∠CAB=40^0. Prove that: $a^3-b^3=3ab^2$**



Proposed by Jafar Nikpour-Iran

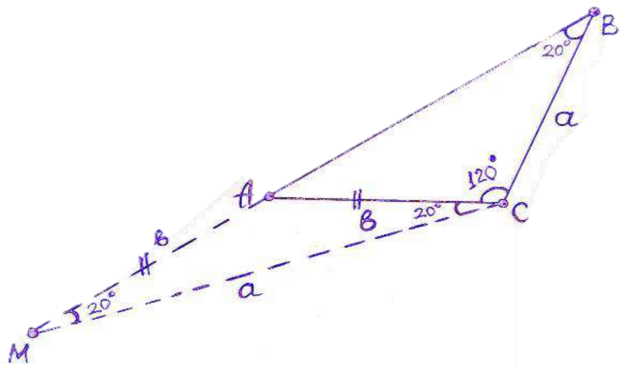
Solution by Togrul Ehmedov-Azerbaijan

Using the law of cosines in △ABC, we conclude that

$$AB=\sqrt{a^2+b^2-2ab\cos(120^0)}=\sqrt{a^2+b^2+ab}$$

$$\triangle BCM\sim\triangle MA,\ \frac{MB}{MC}=\frac{MC}{AC}\Rightarrow\frac{MA+AB}{MC}=\frac{MC}{AC}$$

$$\frac{b+\sqrt{a^2+b^2+ab}}{a}=\frac{a}{b},\qquad b\sqrt{a^2+b^2+ab}=a^2-b^2,\qquad a^3-b^3=3ab^2$$



1559. **In any △ ABC, the following relationship holds :**

$$\sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}}+\left(\frac{R}{2r}\right)^3\geq 1+\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\overset{?}{\geq}32Rr\cos^2\frac{A}{2}\overset{via\ (i)}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)\Leftrightarrow(b+c)^2+4a^2-4a(b+c)\overset{?}{\geq}0$$

$$\Leftrightarrow(b+c-2a)^2\overset{?}{\geq}0\to\text{true}\therefore b+c\geq\sqrt{32Rr}.\cos\frac{A}{2}\ \textit{and analogs}\Rightarrow$$

$$\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\leq\sum_{cyc}\sqrt[2023]{\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}}}=\sqrt[4046]{\frac{R}{2r}}.\sum_{cyc}\sqrt[2023]{\sin\frac{A}{2}}$$

$$\overset{Jensen}{\leq}\sqrt[4046]{\frac{R}{2r}}.3.\sqrt[2023]{\frac{1}{2}}$$

$$\left(\because f''(x)=-\frac{2023\sin^2\frac{A}{2}+2022\cos^2\frac{A}{2}}{16370116\left(\sin\frac{A}{2}\right)^{\frac{4045}{2023}}}<0\text{ where }f(x)=\sqrt[2023]{\sin\frac{x}{2}}\ \forall x\in(0,\pi)\right)$$

$$\therefore\sum_{cyc}\sqrt[2023]{\frac{a}{b+c}}\leq3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{R}{2r}}\to(1)$$

Implementing (1) on *a* triangle with sides $\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$ whose *area as a* consequence of trivi*al* c*alculations* $=\frac{F}{3}$, we *arrive at*: $\sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}\leq$

$$3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{\left(\frac{\frac{2m_a}{3}.\frac{2m_b}{3}.\frac{2m_c}{3}}{\frac{4F}{3}}\right)}{\left(\frac{\frac{2F}{3}}{\frac{m_a}{3}+\frac{m_b}{3}+\frac{m_c}{3}}\right)}}\left(\because\frac{R}{2r}=\frac{\left(\frac{abc}{4F}\right)}{\left(\frac{2F}{s}\right)}\right)=3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{m_am_bm_c\left(\sum_{cyc}m_a\right)}{9F^2}}$$

$$\overset{\substack{m_am_bm_c\leq\frac{Rs^2}{2}\\ and\\ \text{Leuenberger + Euler}}}{\leq}3.\sqrt[2023]{\frac{1}{2}}.\sqrt[4046]{\frac{\frac{Rs^2}{2}\left(\frac{9R}{2}\right)}{9r^2s^2}}$$

$$\therefore \sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}} \le 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[2023]{\frac{R}{2r}} \to (2)$$

$$\text{Again}, \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} \overset{A-G}{\ge} 3.\sqrt[6069]{\frac{bc.ca.ab}{abc(a+b)(b+c)(c+a)}}$$

$$= 3.\sqrt[6069]{\frac{4Rrs}{2s(s^2+2Rr+r^2)}} \overset{Gerretsen}{\ge} 3.\sqrt[6069]{\frac{2Rr}{4R^2+6Rr+4r^2}} \overset{Euler}{\ge} 3.\sqrt[6069]{\frac{2Rr}{4R^2+3R^2+R^2}}$$

$$= 3.\sqrt[6069]{\frac{8r}{4R}.\frac{1}{8}} \therefore \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} \ge 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[6069]{\frac{2r}{R}} \to (3) \therefore (2),(3) \Rightarrow \text{in order}$$

$$\text{to prove}: \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} + \left(\frac{R}{2r}\right)^3 - 1 \ge \sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}, \text{it suffices to prove}:$$

$$3.\sqrt[2023]{\frac{1}{2}}.\sqrt[6069]{\frac{2r}{R}} + \left(\frac{R}{2r}\right)^3 - 1 \ge 3.\sqrt[2023]{\frac{1}{2}}.\sqrt[2023]{\frac{R}{2r}}$$

$$\Leftrightarrow \left(\frac{R}{2r}\right)^3 - 1 \ge 3.\sqrt[2023]{\frac{1}{2}}.\left(\sqrt[2023]{\frac{R}{2r}} - \sqrt[6069]{\frac{2r}{R}}\right) \text{ and to prove it, it sufices to prove}:$$

$$\left(\frac{R}{2r}\right)^3 - 1 \ge 3\left(\sqrt[2023]{\frac{R}{2r}} - \sqrt[6069]{\frac{2r}{R}}\right)\left(\because \sqrt[2023]{\frac{1}{2}} < 1\right)$$

$$\Leftrightarrow t^{18207} - 1 \ge 3\left(t^3 - \frac{1}{t}\right)\left(\text{where } t = \sqrt[6069]{\frac{R}{2r}} \ge 1\right) \Leftrightarrow t^{18208} - t \overset{(*)}{\ge} 3(t^4-1)$$

$$\text{Let } f(t) = t^{18208} - t - 3t^4 + 3\ \forall\, t \ge 1 \text{ and then}: f'(t) = 18208t^{18207} - 12t^3 - 1$$

$$= 12t^3(t^{18204}-1) + (t^{18207}-1) \overset{t\ge 1}{\ge} 0 \Rightarrow f(t) \text{ is } \uparrow \forall\, t \ge 1 \Rightarrow f(t) \ge f(1) = 0$$

$$\Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\sqrt[2023]{\frac{h_a}{h_b+h_c}} + \left(\frac{R}{2r}\right)^3 \ge 1 + \sum_{cyc}\sqrt[2023]{\frac{m_a}{m_b+m_c}}$$

$\forall\, \Delta\, ABC$, " = " iff $\Delta\, ABC$ is equilateral (QED)

Proof of $m_a m_b m_c \le \frac{Rs^2}{2}$

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc}a^6 + 6\left(\sum_{cyc}a^4b^2 + \sum_{cyc}a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now}, \sum_{cyc}a^6 = \left(\sum_{cyc}a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$=\left(\sum_{cyc}a^2\right)^3-3\left(2a^2b^2c^2+\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\right)$$

$$=\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\therefore\sum_{cyc}a^6\overset{(2)}{=}\left(\sum_{cyc}a^2\right)^3+3a^2b^2c^2-3\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)$$

$$\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4=\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)\overset{(3)}{=}$$

$$\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-3a^2b^2c^2\therefore(1),(2),(3)\Rightarrow m_a^2m_b^2m_c^2$$

$$=\frac{1}{64}\left(\begin{matrix}-4\left(\sum_{cyc}a^2\right)^3-12a^2b^2c^2+12\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)\\+6\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-18a^2b^2c^2+3a^2b^2c^2\end{matrix}\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left(-4\left(\sum_{cyc}a^2\right)^3+18\left(\left(\sum_{cyc}ab\right)^2-16Rrs^2\right)\left(\sum_{cyc}a^2\right)-27a^2b^2c^2\right)$$

$$=\frac{1}{64}\left\{\begin{matrix}-32(s^2-4Rr-r^2)^3+36(s^2-4Rr-r^2)(s^2+4Rr+r^2)^2\\-576Rrs^2(s^2-4Rr-r^2)-432R^2r^2s^2\end{matrix}\right\}$$

$$=\frac{1}{16}\{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\}$$

$$\le\frac{R^2s^4}{4}\Leftrightarrow$$

$$s^6-s^4(4R^2+12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3\overset{(\bullet)}{\le}0$$

Now, LHS of $(\bullet)\overset{\text{Gerretsen}}{\le}-s^4(8Rr-36r^2)-s^2(60R^2r^2+120Rr^3+33r^4)$

$$-r^3(4R+r)^3\overset{?}{\le}0$$

$$\Leftrightarrow s^4(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(a)} s^2(16Rr-5r^2)(8R-16r)$$

$$+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3 \text{ and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(b)} 20rs^2(4R^2+4Rr+3r^2)$$

$(a),(b) \Rightarrow$ **in order to prove** $(\bullet\bullet)$, **it suffices to prove :**

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$
$$\geq 20rs^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3 \geq 0$$
$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\geq} 27r^2s^2$$

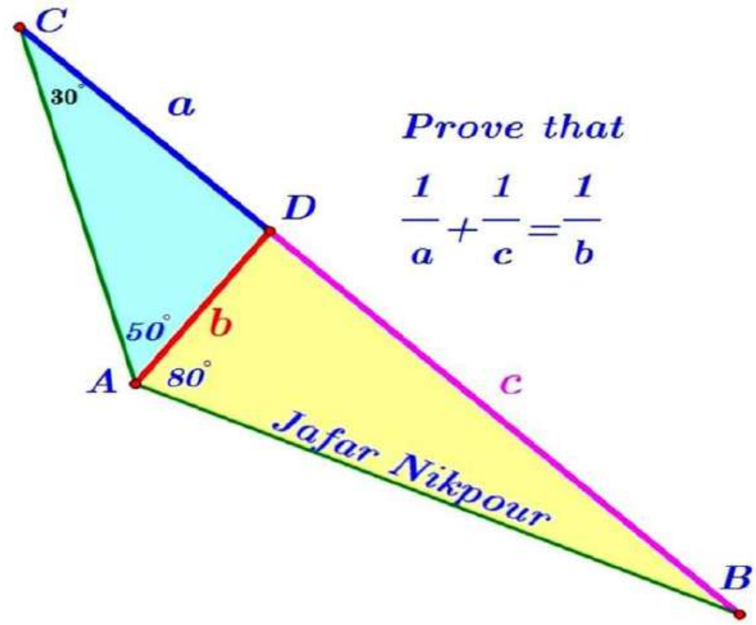
$$\text{Now, LHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\geq}}_{(c)} (108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$$

$$\text{and RHS of } (\bullet\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\leq}}_{(d)} 27r^2(4R^2+4Rr+3r^2)$$

$(c),(d) \Rightarrow$ **in order to prove** $(\bullet\bullet\bullet)$, **it suffices to prove :**

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3 \geq 27r^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 224t^3-587t^2+308t-60 \geq 0 \left(\text{where } t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet)$$
$$\Rightarrow (\bullet) \text{ is true} \Rightarrow m_a^2m_b^2m_c^2 \leq \frac{R^2s^4}{4} \Rightarrow m_am_bm_c \leq \frac{Rs^2}{2} \text{ (QED)}$$

1560.



Proposed by Jafar Nikpour-Iran

Solution by Daniel Sitaru-Romania

$$m(\sphericalangle CDA) = 180° - 50° - 30° = 100°, m(\sphericalangle ADB) = 180° - 100° = 80°$$

In ΔADC:

$$\frac{a}{\sin 50°} = \frac{b}{\sin 30°} \Rightarrow b = \frac{a\sin 30°}{\sin 50°}$$

In ΔADB:

$$\frac{c}{\sin 80°} = \frac{b}{\sin 20°} \Rightarrow c = \frac{b\sin 80°}{\sin 20°} = \frac{a\sin 30°}{\sin 50°} \cdot \frac{\sin 80°}{\sin 20°}$$

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} \Leftrightarrow \frac{1}{a} + \frac{1}{\frac{a\sin 30°}{\sin 50°} \cdot \frac{\sin 80°}{\sin 20°}} = \frac{1}{\frac{a\sin 30°}{\sin 50°}} \Leftrightarrow 1 + \frac{\sin 20° \sin 50°}{\sin 30° \cdot \sin 80°} = \frac{\sin 50°}{\sin 30°} \Leftrightarrow$$

$$\Leftrightarrow \sin 30° \cdot \sin 80° + \sin 20° \sin 50° = \sin 50° \sin 80° \Leftrightarrow$$

$$\Leftrightarrow \sin 20° \sin 50° = \sin 80°(\sin 50° - \sin 30°) \Leftrightarrow$$

$$\Leftrightarrow 2\sin 10° \cos 10° \sin 50° = \sin(90° - 10°) \cdot 2\sin\frac{50° - 30°}{2} \cos\frac{50° + 30°}{2} \Leftrightarrow$$

$$\Leftrightarrow 2\sin 10° \cos 10° \sin 50° = 2\cos 10° \sin 10° \cos 40° \Leftrightarrow$$

$$\Leftrightarrow \sin 50° = \cos 40° \Leftrightarrow \sin 50° = \sin 50°$$

1561. In ΔABC the following relationship holds:

$$\frac{r_a^3(r_a^2 + r_b r_c)}{r_b^4 + r_c^4} + \frac{r_b^3(r_b^2 + r_c r_a)}{r_c^4 + r_a^4} + \frac{r_c^3(r_c^2 + r_a r_b)}{r_a^4 + r_b^4} \geq 9r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

WLOG: $a \leq b \leq c$.

$$-a \geq -b \geq -c \Rightarrow s - a \geq s - b \geq s - c \Rightarrow \frac{1}{s-a} \leq \frac{1}{s-b} \leq \frac{1}{s-c}$$

$$\frac{F}{s-a} \leq \frac{F}{s-b} \leq \frac{F}{s-c} \Rightarrow r_a \leq r_b \leq r_c$$

$$\sum_{cyc} \frac{r_a^3(r_a^2 + r_b r_c)}{r_b^4 + r_c^4} \geq \sum_{cyc} \frac{r_a^3(r_a^2 + r_a r_a)}{r_b^4 + r_c^4} = \sum_{cyc} \frac{r_a^5 + r_a^5}{r_b^4 + r_c^4} \geq$$

$$\overset{LEHMER}{\geq}\sum_{cyc}\frac{r_a^4+r_a^4}{r_b^3+r_c^3}\overset{LEHMER}{\geq}\sum_{cyc}\frac{r_a^3+r_a^3}{r_b^2+r_c^2}\overset{LEHMER}{\geq}\sum_{cyc}\frac{r_a^2+r_a^2}{r_b+r_c}\overset{LEHMER}{\geq}$$

$$\geq\sum_{cyc}\frac{r_b+r_c}{1+1}=\sum_{cyc}r_a\overset{AM-GM}{\geq}3\sqrt[3]{r_ar_br_c}=$$

$$=3\sqrt[3]{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}=3\sqrt[3]{\frac{sF^3}{s(s-a)(s-b)(s-c)}}=$$

$$=3\sqrt[3]{\frac{sF^3}{F^2}}=3\sqrt[3]{sF}=3\sqrt[3]{rs^2}\overset{MITRINOVIC}{\geq}3\sqrt[3]{r\left(3\sqrt{3}r\right)^2}=$$

$$=3\sqrt[3]{27r^3}=3\cdot 3r=9r$$

Equality holds for: $a=b=c$.

1562.

In any Δ ABC, the following relationship holds :

$$\frac{h_a(w_b^2+m_c^2)}{h_a^2+w_bm_c}+\frac{w_b(m_c^2+h_a^2)}{w_b^2+m_ch_a}+\frac{m_c(h_a^2+w_b^2)}{m_c^2+h_aw_b}\geq 9r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

Firstly, we shall prove that $\forall\, x,y,z>0$, $\prod_{cyc}(y^2+z^2)\geq\prod_{cyc}(x^2+yz)$

$$\Leftrightarrow\sum_{cyc}x^4y^2+\sum_{cyc}x^2y^4\overset{(i)}{\geq}xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3$$

$$\text{LHS of (i)}=\sum_{cyc}\frac{x^4y^2+x^4z^2}{2}+\sum_{cyc}\frac{x^4y^2+x^2y^4}{2}\overset{A-G}{\geq}\sum_{cyc}x^4yz+\sum_{cyc}x^3y^3$$

$$=xyz\sum_{cyc}x^3+\sum_{cyc}x^3y^3\Rightarrow\text{(i) is true}\therefore\forall\, x,y,z>0,\ \frac{\prod_{cyc}(y^2+z^2)}{\prod_{cyc}(x^2+yz)}\geq 1\rightarrow(1)$$

$$\text{Now, }\frac{h_a(w_b^2+m_c^2)}{h_a^2+w_bm_c}+\frac{w_b(m_c^2+h_a^2)}{w_b^2+m_ch_a}+\frac{m_c(h_a^2+w_b^2)}{m_c^2+h_aw_b}$$

$$\overset{A-G}{\geq}3\sqrt[3]{h_aw_bm_c\cdot\frac{(w_b^2+m_c^2)(m_c^2+h_a^2)(h_a^2+w_b^2)}{(h_a^2+w_bm_c)(w_b^2+m_ch_a)(m_c^2+h_aw_b)}}$$

$$\geq 3.\sqrt[3]{h_a h_b h_c.\frac{(y^2+z^2)(z^2+x^2)(x^2+y^2)}{(x^2+yz)(y^2+zx)(z^2+xy)}} \overset{via\ (1)}{\geq} 3.\sqrt[3]{\frac{2r^2s^2}{R}}$$

$$\overset{Gerretsen}{\geq} 3.\sqrt[3]{\frac{r^2.(27Rr+5r(R-2r))}{R}} \overset{Euler}{\geq} 3.\sqrt[3]{\frac{r^2.27Rr}{R}} = 9r$$

$$\therefore \frac{h_a(w_b^2+m_c^2)}{h_a^2+w_b m_c} + \frac{w_b(m_c^2+h_a^2)}{w_b^2+m_c h_a} + \frac{m_c(h_a^2+w_b^2)}{m_c^2+h_a w_b} \geq 9r\ \forall\ \Delta\ ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)

1563.

In any Δ ABC, the following relationship holds :

$$\frac{h_a^2(h_a^2+w_b m_c)}{(w_b+m_c)^2} + \frac{w_b^2(w_b^2+m_c h_a)}{(m_c+h_a)^2} + \frac{m_c^2(m_c^2+h_a w_b)}{(h_a+w_b)^2} \geq \frac{27r^2}{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $h_a = x, w_b = y$ and $m_c = z$ and then :

$$\frac{h_a^2(h_a^2+w_b m_c)}{(w_b+m_c)^2} + \frac{w_b^2(w_b^2+m_c h_a)}{(m_c+h_a)^2} + \frac{m_c^2(m_c^2+h_a w_b)}{(h_a+w_b)^2}$$

$$= \sum_{cyc}\frac{x^2(x^2+yz)}{(y+z)^2} = \sum_{cyc}\frac{x^4}{(y+z)^2} + xyz\sum_{cyc}\frac{x}{(y+z)^2}$$

$$\overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}\frac{x^2}{(y+z)^2}\right) + \frac{xyz}{3}\left(\sum_{cyc}\frac{1}{y+z}\right)\left(\sum_{cyc}\frac{x}{y+z}\right)$$

$$\left(\begin{array}{c} \because \text{WLOG assuming } x \geq y \geq z \Rightarrow x^2 \geq y^2 \geq z^2, \frac{x^2}{(y+z)^2} \geq \frac{y^2}{(z+x)^2} \geq \frac{z^2}{(x+y)^2}, \\ \frac{1}{y+z} \geq \frac{1}{z+x} \geq \frac{1}{x+y} \text{ and } \frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y} \end{array}\right)$$

$$\overset{\substack{Nesbitt\\ and\\ Bergstrom}}{\geq} \frac{1}{3}\left(\sum_{cyc}x^2\right)\frac{1}{3}\left(\sum_{cyc}\frac{x}{y+z}\right)^2 + \frac{xyz}{3}.\frac{9}{2\sum_{cyc}x}.\frac{3}{2} \overset{Nesbitt}{\geq}$$

$$\frac{1}{9}\left(\sum_{cyc}x^2\right).\frac{9}{4} + \frac{9xyz}{4\sum_{cyc}x} \overset{?}{\geq} \frac{3}{2}.\sqrt[3]{x^2y^2z^2} \Leftrightarrow \boxed{\frac{\sum_{cyc}x^2}{2} + \frac{9xyz}{2\sum_{cyc}x} \underset{(*)}{\overset{?}{\geq}} 3.\sqrt[3]{x^2y^2z^2}}$$

Assigning $y+z = X, z+x = Y, x+y = Z \Rightarrow X+Y-Z = 2z > 0, Y+Z-X = 2x > 0$ and $Z+X-Y = 2y > 0 \Rightarrow X+Y > Z, Y+Z > X, Z+X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (1) \Rightarrow x = s - X, y = s - Y,$$

$$z = s - Z \text{ and such substitutions} \Rightarrow xyz = (s - X)(s - Y)(s - Z)$$

$$\Rightarrow xyz = r^2 s \to (2); \sum_{cyc} xy = \sum_{cyc} (s - X)(s - Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (3)$$

$$\text{and } \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \to (4) \therefore \text{via (1), (2) and (4)}, (*) \Leftrightarrow$$

$$\frac{s^2 - 8Rr - 2r^2}{2} + \frac{9r^2 s}{2s} \geq 3.\sqrt[3]{r^4 s^2} \Leftrightarrow \boxed{(s^2 - 8Rr + 7r^2)^3 - 216r^4 s^2 \overset{(**)}{\geq} 0}$$

$$\text{and} \because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$(s^2 - 8Rr + 7r^2)^3 - 216r^4 s^2 \geq (s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (24Rr + 6r^2)s^4 - r^2 s^2(576R^2 - 144Rr + 144r^2)$$

$$+r^3(3584R^3 - 2496R^2 r + 24Rr^2 + 218r^3) \overset{(***)}{\geq} 0 \text{ and}$$

$$\because (24Rr + 6r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (***) \geq (24Rr + 6r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow \boxed{(48R^2 + 24Rr - 51r^2)s^2 \overset{(****)}{\geq} r(640R^3 + 48R^2 r - 96Rr^2 - 17r^3)}$$

$$\text{Now, LHS of } (****) \overset{\text{Gerretsen}}{\geq} (48R^2 + 24Rr - 51r^2)(16Rr - 5r^2) \overset{?}{\geq}$$

$$r(640R^3 + 48R^2 r - 96Rr^2 - 17r^3) \Leftrightarrow 16t^3 + 12t^2 - 105t + 34 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)(16t^2 + 44t - 17) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***) \Rightarrow (**)$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{h_a^2(h_a^2 + w_b m_c)}{(w_b + m_c)^2} + \frac{w_b^2(w_b^2 + m_c h_a)}{(m_c + h_a)^2} + \frac{m_c^2(m_c^2 + h_a w_b)}{(h_a + w_b)^2}$$

$$\geq \frac{3}{2}.\sqrt[3]{h_a^2 w_b^2 m_c^2} \geq \frac{3}{2}.\sqrt[3]{h_a^2 h_b^2 h_c^2} = \frac{3}{2}, \left(\sqrt[3]{\frac{2r^2 s^2}{R}}\right)^2 \overset{\text{Gerretsen}}{\geq}$$

$$\frac{3}{2}.\left(\sqrt[3]{\frac{r^2.(27Rr + 5r(R - 2r))}{R}}\right)^2 \overset{\text{Euler}}{\geq} \frac{3}{2}.\left(\sqrt[3]{\frac{r^2.27Rr}{R}}\right)^2$$

$$\therefore \frac{h_a^2(h_a^2 + w_b m_c)}{(w_b + m_c)^2} + \frac{w_b^2(w_b^2 + m_c h_a)}{(m_c + h_a)^2} + \frac{m_c^2(m_c^2 + h_a w_b)}{(h_a + w_b)^2} \geq \frac{27r^2}{2},$$

$$\forall \Delta ABC,'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x, y, z > 0$. We have

$$\sum_{cyc}\frac{x^2(x^2+yz)}{(y+z)^2}$$

$$=\sum_{cyc}\frac{(x^2)^3}{(xy+zx)^2}+xyz.\sum_{cyc}\frac{x^3}{(xy+zx)^2}\overset{Hölder}{\geq}\frac{\left(\sum_{cyc}x^2\right)^3}{4\left(\sum_{cyc}yz\right)^2}$$

$$+xyz.\frac{\left(\sum_{cyc}x\right)^3}{4\left(\sum_{cyc}yz\right)^2}$$

$$\underset{\left(\sum_{cyc}x\right)^2\geq 3\sum_{cyc}yz}{\overset{\sum_{cyc}x^2\geq\sum_{cyc}yz}{\geq}}\frac{\sum_{cyc}yz}{4}+\frac{3xyz\sum_{cyc}x}{4\sum_{cyc}yz}\overset{AM-GM}{\geq}\frac{1}{2}\sqrt{3xyz(x+y+z)}.$$

Setting $x=h_a,\ \ y=w_b,\ \ z=m_c$, we obtain

$$\frac{{h_a}^2({h_a}^2+w_bm_c)}{(w_b+m_c)^2}+\frac{{w_b}^2({w_b}^2+m_ch_a)}{(m_c+h_a)^2}+\frac{{m_c}^2({m_c}^2+h_aw_b)}{(h_a+w_b)^2}$$

$$\geq\frac{1}{2}\sqrt{3h_aw_bm_c(h_a+w_b+m_c)}$$

$$\underset{m_c\geq h_c}{\overset{w_b\geq h_b}{\geq}}\frac{3}{2}\sqrt{h_ah_bh_c.\frac{h_a+h_b+h_c}{3}}\underset{AM-HM}{\overset{GM-HM}{\geq}}\frac{3}{2}\left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^2=\frac{3}{2}\left(\frac{3}{\frac{1}{r}}\right)^2=\frac{27r^2}{2},$$

as desired. Equality holds iff ΔABC is equilateral.

1564. In ΔABC the following relationship holds:

$$\frac{r_a^n(r_a^2+r_br_c)}{r_b^{n+1}+r_c^{n+1}}+\frac{r_b^n(r_b^2+r_cr_a)}{r_c^{n+1}+r_a^{n+1}}+\frac{r_c^n(r_c^2+r_ar_b)}{r_a^{n+1}+r_b^{n+1}}\geq 9r, n\in\mathbb{N}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

WLOG: $a\leq b\leq c$.

$$-a\geq -b\geq -c\Rightarrow s-a\geq s-b\geq s-c\Rightarrow\frac{1}{s-a}\leq\frac{1}{s-b}\leq\frac{1}{s-c}$$

$$\frac{F}{s-a}\leq\frac{F}{s-b}\leq\frac{F}{s-c}\Rightarrow r_a\leq r_b\leq r_c$$

$$\sum_{cyc}\frac{r_a^n(r_a^2+r_br_c)}{r_b^{n+1}+r_c^{n+1}}\geq\sum_{cyc}\frac{r_a^n(r_a^2+r_ar_a)}{r_b^{n+1}+r_c^{n+1}}=\sum_{cyc}\frac{r_a^{n+2}+r_a^{n+2}}{r_b^{n+1}+r_c^{n+1}}\geq$$

$$\overset{LEHMER}{\geq}\sum_{cyc}\frac{r_a^{n+1}+r_a^{n+1}}{r_b^n+r_c^n}\overset{LEHMER}{\geq}\sum_{cyc}\frac{r_a^n+r_a^n}{r_b^{n-1}+r_c^{n-1}}\overset{LEHMER}{\geq}\ldots\geq\sum_{cyc}\frac{r_a^2+r_a^2}{r_b+r_c}\overset{LEHMER}{\geq}$$

$$\geq \sum_{cyc} \frac{r_b + r_c}{1+1} = \sum_{cyc} r_a \overset{AM-GM}{\geq} 3\sqrt[3]{r_a r_b r_c} =$$

$$= 3\sqrt[3]{\frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}} = 3\sqrt[3]{\frac{sF^3}{s(s-a)(s-b)(s-c)}} =$$

$$= 3\sqrt[3]{\frac{sF^3}{F^2}} = 3\sqrt[3]{sF} = 3\sqrt[3]{rs^2} \overset{MITRINOVIC}{\geq} 3\sqrt[3]{r\left(3\sqrt{3}r\right)^2} =$$

$$= 3\sqrt[3]{27r^3} = 3 \cdot 3r = 9r$$

Equality holds for: $a = b = c$.

1565. **In ΔABC the following relationship holds:**

$$\frac{r_a^2 + r_b r_c}{r_a^4(r_b + r_c)} + \frac{r_b^2 + r_c r_a}{r_b^4(r_c + r_a)} + \frac{r_c^2 + r_a r_b}{r_c^4(r_a + r_b)} \geq \frac{16r}{9R^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

WLOG: $a \leq b \leq c$.

$$-a \geq -b \geq -c \Rightarrow s-a \geq s-b \geq s-c \Rightarrow \frac{1}{s-a} \leq \frac{1}{s-b} \leq \frac{1}{s-c}$$

$$\frac{F}{s-a} \leq \frac{F}{s-b} \leq \frac{F}{s-c} \Rightarrow r_a \leq r_b \leq r_c$$

$$\sum_{cyc} \frac{r_a^2 + r_b r_c}{r_a^4(r_b + r_c)} \geq \sum_{cyc} \frac{r_a^2 + r_a r_a}{r_a^4(r_b + r_c)} = \sum_{cyc} \frac{2r_a^2}{r_a^4(r_b + r_c)} =$$

$$= 2\sum_{cyc} \frac{\frac{1}{r_a^2}}{r_b + r_c} \overset{BERGSTROM}{\geq} 2 \cdot \frac{\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right)^2}{2(r_a + r_b + r_c)} =$$

$$= \frac{\frac{1}{r^2}}{4R + r} = \frac{r}{r^3(4R + r)} \overset{EULER}{\geq} \frac{r}{\left(\frac{R}{2}\right)^3\left(4R + \frac{R}{2}\right)} = \frac{16r}{9R^4}$$

Equality holds for: $a = b = c$.

1566. In any Δ ABC, the following relationship holds :

$$\sqrt[3]{\frac{3R}{r}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)^2} \le \frac{4(R-r)}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2$$

$$= as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \Rightarrow n_a^2 = s^2 - 2h_ar_a$$

$$\text{and analogs} \therefore \sum_{cyc}\frac{n_a^2}{r_a^2} = \sum_{cyc}\frac{s^2-2h_ar_a}{r_a^2} = \frac{s^2\sum_{cyc} r_b^2r_c^2}{r^2s^4} - 2.\sum_{cyc}\frac{2rs}{4R\cos\frac{A}{2}\sin\frac{A}{2}.s\tan\frac{A}{2}}$$

$$= \frac{s^4 - 2rs^2(4R+r)}{r^2s^2} - \frac{r}{R}\sum_{cyc}\frac{bc(s-a)}{r^2s} = \frac{s^2-8Rr-2r^2}{r^2} - \frac{s^2-8Rr+r^2}{Rr}$$

$$= \frac{R(s^2-8Rr-2r^2) - r(s^2-8Rr+r^2)}{Rr^2}$$

$$= \frac{(R-r)s^2 - R(8Rr+2r^2) + r(8Rr-r^2)}{Rr^2}$$

$$\overset{\text{Gerretsen}}{\le} \frac{(R-r)(4R^2+4Rr+3r^2) - R(8Rr+2r^2) + r(8Rr-r^2)}{Rr^2}$$

$$\therefore \sum_{cyc}\frac{n_a^2}{r_a^2} \le \frac{4R^3 - 8R^2r + 5Rr^2 - 4r^3}{Rr^2} \to (1)$$

$$\text{Now, } \sqrt[3]{\frac{3R}{r}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)^2} \overset{CBS}{\le} \sqrt[3]{\frac{9R}{r}\sum_{cyc}\frac{n_a^2}{r_a^2}} \overset{\text{via }(1)}{\le} \sqrt[3]{\frac{9R}{r}.\frac{4R^3-8R^2r+5Rr^2-4r^3}{Rr^2}} \overset{?}{<} \frac{4(R-r)}{r}$$

$$\Leftrightarrow 64(R-r)^3 \overset{?}{>} 9(4R^3 - 8R^2r + 5Rr^2 - 4r^3)$$

$$\Leftrightarrow 28t^3 - 120t^2 + 147t - 28 \overset{?}{>} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)((t-2)(28t-8)+3) + 10 \overset{?}{>} 0$$

$$\therefore \sqrt[3]{\frac{3R}{r}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)^2} < \frac{4(R-r)}{r} \quad \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{3R}{r}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)^2 \le \frac{3R}{r}\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\left(\frac{{n_a}^2}{r_a}+\frac{{n_b}^2}{r_b}+\frac{{n_c}^2}{r_c}\right)$$

$$=\frac{3R}{r^2}\sum_{cyc}\frac{s-a}{F}.s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$=\frac{3R}{r^3}\sum_{cyc}(s-a)\left(s-\frac{4(s-b)(s-c)}{a}\right)=\frac{3R}{r^3}\left(s^2-4sr^2.\frac{s^2+r^2+4Rr}{4Rsr}\right)$$

$$=\frac{3[(R-r)s^2-r^2(4R+r)]}{r^3}\overset{Gerretsen}{\le}\frac{3[(R-r)(4R^2+4Rr+3r^2)-r^2(4R+r)]}{r^3}$$

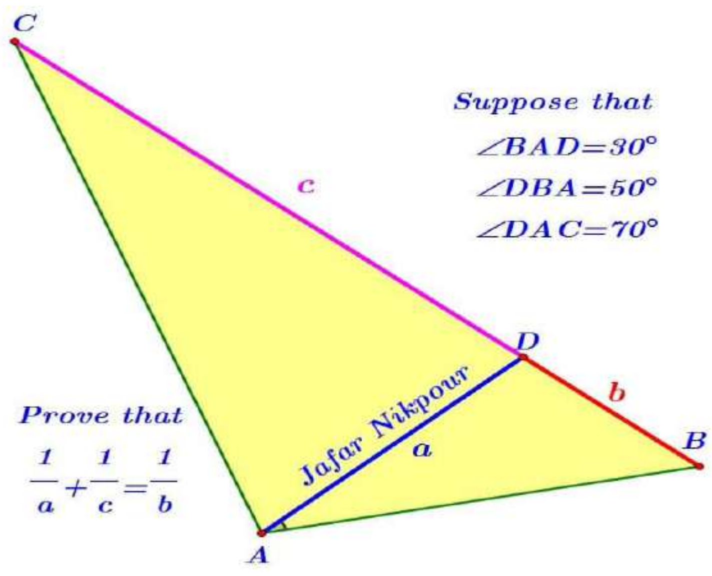
$$=\frac{3(4R^3-5Rr^2-4r^3)}{r^3}$$

$$=\frac{54(R-r)^3-(R-2r)(42R^2-78Rr+21r^2)}{r^3}\overset{Euler}{\le}\frac{54(R-r)^3}{r^3}$$

Therefore

$$\sqrt[3]{\frac{3R}{r}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)^2}\le\frac{3\sqrt[3]{2}(R-r)}{r}\le\frac{4(R-r)}{r}.$$

1567.



Proposed by Jafar Nikpour-Iran

Solution by Daniel Sitaru-Romania

In ΔADB: $m(\measuredangle ADB)=180°-30°-50°=100°$

In ΔADC: $m(\measuredangle ADC)=180°-100°=80°$, $m(\measuredangle DCA)=180°-80°-70°=30°$

In ΔADB:

$$\frac{a}{\sin 50^\circ}=\frac{b}{\sin 30^\circ}\Rightarrow b=\frac{a\sin 30^\circ}{\sin 50^\circ}$$

In ΔADC:

$$\frac{a}{\sin 30^\circ}=\frac{c}{\sin 70^\circ}\Rightarrow c=\frac{a\sin 70^\circ}{\sin 30^\circ}$$

$$\frac{1}{a}+\frac{1}{c}=\frac{1}{b}\Leftrightarrow\frac{1}{a}+\frac{1}{\frac{a\sin 70^\circ}{\sin 30^\circ}}=\frac{1}{\frac{a\sin 30^\circ}{\sin 50^\circ}}\Leftrightarrow 1+\frac{\sin 30^\circ}{\sin 70^\circ}=\frac{\sin 50^\circ}{\sin 30^\circ}$$

$$\sin 70^\circ\cdot\sin 30^\circ+\sin 30^\circ\cdot\sin 30^\circ=\sin 50^\circ\cdot\sin 70^\circ$$

$$\sin 30^\circ(\sin 70^\circ+\sin 30^\circ)=\sin 50^\circ\cdot\sin(90^\circ-20^\circ)$$

$$\frac{1}{2}\cdot 2\sin\frac{70^\circ+30^\circ}{2}\cos\frac{70^\circ+30^\circ}{2}=\sin 50^\circ\cos 20^\circ$$

$$\sin 50^\circ\cos 20^\circ=\sin 50^\circ\cos 20^\circ$$

1568. In any Δ ABC, the following relationship holds :

$$\frac{8R}{r}\le\sum_{cyc}\frac{(b+c)^2}{r_a^2}\le 16\left(\frac{R^2}{r^2}-3\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\ge b\ge c$ and then :

$$(b+c)^2\le(c+a)^2\le(a+b)^2 \text{ and } \frac{1}{r_a^2}\le\frac{1}{r_b^2}\le\frac{1}{r_c^2}\ \therefore \text{ via Chebyshev,}$$

$$\sum_{cyc}\frac{(b+c)^2}{r_a^2}\ge\left(\sum_{cyc}(b+c)^2\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right)$$

$$\overset{A-G}{\ge} 4\left(\sum_{cyc}ab\right).\frac{\left(\sum_{cyc}r_br_c\right)^2-2r_ar_br_c\sum_{cyc}r_a}{r^2s^4}$$

$$=4(s^2+4Rr+r^2).\frac{s^4-2rs^2(4R+r)}{r^2s^4}\overset{?}{\ge}\frac{8R}{r}$$

$$\Leftrightarrow s^4-(10Rr+r^2)s^2-2r^2(4R+r)^2\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)\overset{Gerretsen}{\ge}(6Rr-6r^2)s^2-2r^2(4R+r)^2\overset{Gerretsen}{\ge}$$

$$(6Rr-6r^2)(16Rr-5r^2)-2r^2(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow 32R^2-71Rr+14r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(32R-7r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\frac{(b+c)^2}{r_a^2} \geq \frac{8R}{r}$$

$$\text{Again}, \sum_{cyc}\left((b+c)^2(s-a)^2\right) = \sum_{cyc}\left((s+s-a)^2(s-a)^2\right)$$

$$= \sum_{cyc}\left(\left(s^2+(s-a)^2+2s(s-a)\right)(s-a)^2\right)$$

$$= s^2\sum_{cyc}(s^2-2sa+a^2)+2s\sum_{cyc}(s^3-3s^2a+3sa^2-a^3)$$

$$+\sum_{cyc}(s^4-4s^3a+6s^2a^2-4sa^3+a^4)$$

$$= s^2.3s^2-2s^3.2s+s^2\sum_{cyc}a^2+2s.3s^3-6s^3.2s+6s^2\sum_{cyc}a^2-2s\sum_{cyc}a^3+3s^4$$

$$-4s^3.2s+6s^2\sum_{cyc}a^2-4s\sum_{cyc}a^3+2\sum_{cyc}a^2b^2-16r^2s^2$$

$$= -12s^4+26s^2(s^2-4Rr-r^2)-12s^2(s^2-6Rr-3r^2)$$

$$+2\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-16r^2s^2$$

$$= 2\left(2s^4-(24Rr+r^2)s^2+r^2(4R+r)^2\right)$$

$$\Rightarrow \sum_{cyc}\frac{(b+c)^2}{r_a^2} = \frac{2\left(2s^4-(24Rr+r^2)s^2+r^2(4R+r)^2\right)}{r^2s^2} \overset{?}{\leq} 16\left(\frac{R^2}{r^2}-3\right)$$

$$\Leftrightarrow 2s^4+r^2(4R+r)^2 \underset{(**)}{\overset{?}{\leq}} (8R^2+24Rr-23r^2)s^2$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\leq} (8R^2+8Rr+6r^2)s^2+r^2(4R+r)^2 \overset{?}{\leq}$$

$$(8R^2+24Rr-23r^2)s^2 \Leftrightarrow (16Rr-29r^2)s^2 \underset{(***)}{\overset{?}{\geq}} r^2(4R+r)^2$$

$$\text{Again, LHS of } (***) \overset{\text{Gerretsen}}{\geq} (16Rr-29r^2)(16Rr-5r^2) \overset{?}{\geq} r^2(4R+r)^2$$

$$\Leftrightarrow 240R^2-552Rr+144r^2 \overset{?}{\geq} 0 \Leftrightarrow 24(R-2r)(10R-3r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (***) \Rightarrow (**) \text{ is true} \therefore \sum_{cyc}\frac{(b+c)^2}{r_a^2} \leq 16\left(\frac{R^2}{r^2}-3\right) \therefore \frac{8R}{r} \leq \sum_{cyc}\frac{(b+c)^2}{r_a^2}$$

$$\leq 16\left(\frac{R^2}{r^2}-3\right) \forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1569.

In any acute triangle ABC, the following relationship holds :

$$\frac{a}{b+c}.\sqrt{\sin A}+\frac{b}{c+a}.\sqrt{\sin B}+\frac{c}{a+b}.\sqrt{\sin C} > 1$$

Proposed by Vasile Mircea Popa, Mihai Neghină-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Via Power}-\text{Mean}-\text{Inequality},\left(\frac{\sum_{cyc}a^{\frac{3}{2}}}{3}\right)^{\frac{2}{3}}\geq\left(\frac{\sum_{cyc}a^{1}}{3}\right)^{1}$$

$$\Rightarrow\frac{\sum_{cyc}a^{\frac{3}{2}}}{3}\geq\left(\frac{2s}{3}\right)^{\frac{3}{2}}=\frac{2s}{3}.\sqrt{\frac{2s}{3}}\Rightarrow\sum_{cyc}a^{\frac{3}{2}}\geq 2s.\sqrt{\frac{2s}{3}}\rightarrow(1)$$

$$\frac{a}{b+c}.\sqrt{\sin A}+\frac{b}{c+a}.\sqrt{\sin B}+\frac{c}{a+b}.\sqrt{\sin C}=\frac{1}{\sqrt{2R}}.\sum_{cyc}\frac{a\sqrt{a}}{b+c}\overset{\text{Chebyshev}}{\geq}$$

$$\frac{1}{3.\sqrt{2R}}.\left(\sum_{cyc}a\sqrt{a}\right)\left(\frac{1}{b+c}\right)\left(\begin{matrix}\because\text{WLOG assuming } a\geq b\geq c\Rightarrow a\sqrt{a}\geq b\sqrt{b}\geq c\sqrt{c}\\ \text{and }\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}\end{matrix}\right)\overset{\substack{\text{via (1)}\\ \text{and}\\ \text{Bergstrom}}}{\geq}$$

$$\frac{1}{3.\sqrt{2R}}.2s.\sqrt{\frac{2s}{3}}.\frac{9}{4s}=\frac{3}{2.\sqrt{2R}}.\sqrt{\frac{2s}{3}}>\frac{3}{2.\sqrt{2R}}.\sqrt{\frac{4R}{3}}$$

$$(\because\Delta\,ABC\text{ is acute}\Rightarrow s>2R+r>2R)=\sqrt{\frac{3}{2}}>1$$

$$\therefore\frac{a}{b+c}.\sqrt{\sin A}+\frac{b}{c+a}.\sqrt{\sin B}+\frac{c}{a+b}.\sqrt{\sin C}>1\ \forall\text{ acute }\Delta\,ABC\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $a\geq b\geq c$.

$$\text{We have }\frac{a}{b+c}\geq\frac{b}{c+a}\geq\frac{c}{a+b}\ \&\ \sqrt{\sin A}\geq\sqrt{\sin B}\geq\sqrt{\sin C},$$

then by Chebyshev's inequality, we have

$$\frac{a}{b+c}\sqrt{\sin A}+\frac{b}{c+a}\sqrt{\sin B}+\frac{c}{a+b}\sqrt{\sin C}\geq$$
$$\geq\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right).$$

By Nesbitt's inequality, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Also since $0<\sin x<1,\forall x\in\left(0,\frac{\pi}{2}\right)$, and by using Jordan's inequality, we have

$$\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}>\sin A+\sin B+\sin C\geq\frac{2A}{\pi}+\frac{2B}{\pi}+\frac{2C}{\pi}=2.$$

Therefore

$$\frac{a}{b+c}\sqrt{\sin A}+\frac{b}{c+a}\sqrt{\sin B}+\frac{c}{a+b}\sqrt{\sin C}>\frac{1}{3}.\frac{3}{2}.2=1.$$

1570. In any $\Delta\, ABC$ and $\forall\, n \in \mathbb{N}$, the following relationship holds:

$$\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\boxed{\text{Case 1}}\ n=1 \text{ and then : LHS}=\frac{x^2+yz}{x(y+z)}+\frac{y^2+zx}{y(z+x)}+\frac{z^2+xy}{z(x+y)}$$

$$(x=h_a,y=w_b,z=m_c)=\sum_{cyc}\frac{x}{y+z}+\frac{1}{xyz}\sum_{cyc}\frac{y^2z^2}{y+z}\overset{\substack{\text{Nesbitt}\\ \text{and}\\ \text{Bergstrom}}}{\ge}\frac{3}{2}+\frac{\left(\sum_{cyc}xy\right)^2}{2xyz\sum_{cyc}x}$$

$$=\frac{3}{2}+\frac{3}{2}=3\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad(\because n=1)$$

$$\therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\text{for } n=1$$

$$\boxed{\text{Case 2}}\ n=2 \text{ and then : }\boxed{\text{LHS}}=\frac{x^2+yz}{x^2(y+z)}+\frac{y^2+zx}{y^2(z+x)}+\frac{z^2+xy}{z^2(x+y)}=\sum_{cyc}\frac{1}{y+z}$$

$$+xyz\sum_{cyc}\frac{\left(\frac{1}{x}\right)^3}{y+z}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Holder}}}{\ge}\frac{9}{2\sum_{cyc}x}+xyz.\frac{\left(\sum_{cyc}\frac{1}{x}\right)^3}{6\sum_{cyc}x}\boxed{\overset{?}{\ge}\frac{9}{\sum_{cyc}x}}$$

$$\Leftrightarrow xyz.\frac{\left(\sum_{cyc}\frac{1}{x}\right)^3}{6\sum_{cyc}x}\overset{?}{\ge}\frac{9}{2\sum_{cyc}x}\Leftrightarrow\left(\sum_{cyc}xy\right)^3\overset{?}{\ge}27x^2y^2z^2\rightarrow\text{true via AM}-\text{GM}$$

$$\therefore \text{LHS}\ge\frac{9}{\sum_{cyc}x}=\frac{9}{h_a+w_b+m_c}\ge\frac{9}{\sum_{cyc}m_a}\overset{\substack{\text{Leuenberger}\\ \text{and}\\ \text{Euler}}}{\ge}\frac{9}{\frac{9R}{2}}=\frac{2}{R}=\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}$$

$$(n=2)\ \therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\text{for } n=2$$

$$\boxed{\text{Case 3}}\ n=3 \text{ and then : }\boxed{\text{LHS}}=\frac{x^2+yz}{x^3(y+z)}+\frac{y^2+zx}{y^3(z+x)}+\frac{z^2+xy}{z^3(x+y)}=\sum_{cyc}\frac{1}{xy+xz}$$

$$+xyz\sum_{cyc}\frac{\left(\frac{1}{x}\right)^4}{y+z}\overset{\substack{\text{Bergstrom}\\ \text{and}\\ \text{Holder}}}{\ge}\frac{9}{2\sum_{cyc}xy}+xyz.\frac{\left(\sum_{cyc}\frac{1}{x}\right)^4}{9.2\sum_{cyc}x}=\frac{9}{2\sum_{cyc}xy}+\frac{\left(\sum_{cyc}xy\right)^4}{18x^3y^3z^3\sum_{cyc}x}$$

$$\ge\frac{9}{2\sum_{cyc}xy}+\frac{9x^2y^2z^2\left(\sum_{cyc}x\right)^2}{18x^3y^3z^3\sum_{cyc}x}=\frac{9}{2\sum_{cyc}xy}+\frac{\sum_{cyc}x}{2xyz}\boxed{\overset{?}{\ge}\frac{1}{3}\left(\frac{9}{\sum_{cyc}x}\right)^2}$$

$$\Leftrightarrow 9xyz\left(\sum_{cyc}x\right)^2+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^3\overset{?}{\ge}54xyz\sum_{cyc}xy\to\text{true}$$

$$\because 9xyz\left(\sum_{cyc}x\right)^2+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^3\overset{A-G}{\ge}9xyz.3\sum_{cyc}xy+\left(\sum_{cyc}xy\right).27xyz$$

$$=54xyz\sum_{cyc}xy\therefore \text{LHS}\ge\frac{1}{3}\left(\frac{9}{\sum_{cyc}x}\right)^2=\frac{1}{3}\left(\frac{9}{h_a+w_b+m_c}\right)^2\ge\frac{1}{3}\left(\frac{9}{\sum_{cyc}m_a}\right)^2$$

$$\overset{\substack{\text{Leuenberger}\\ and\\ \text{Euler}}}{\ge}\frac{1}{3}\left(\frac{9}{\frac{9R}{2}}\right)^2=\frac{1}{3}\left(\frac{2}{R}\right)^2=\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad(\because n=3)$$

$$\therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\text{for } n=3$$

$\boxed{Case\ 4}$ $n\in\mathbb{N}-\{1,2,3\}$ *and* then : $\boxed{\text{LHS}}=\dfrac{x^2+yz}{x^n(y+z)}+\dfrac{y^2+zx}{y^n(z+x)}+\dfrac{z^2+xy}{z^n(x+y)}$

$$=\sum_{cyc}\frac{\left(\frac{1}{x}\right)^{n-2}}{y+z}+xyz.\sum_{cyc}\frac{\left(\frac{1}{x}\right)^{n+1}}{y+z}\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}\frac{1}{x}\right)^{n-2}}{3^{n-4}.2\sum_{cyc}x}+xyz.\frac{\left(\sum_{cyc}\frac{1}{x}\right)^{n+1}}{3^{n-1}.2\sum_{cyc}x}$$

$$\boxed{\overset{?}{\ge}\frac{1}{3^{n-2}}.\left(\sum_{cyc}\frac{1}{x}\right)^{n-2}.\frac{9}{\sum_{cyc}x}}\Leftrightarrow\frac{9}{2\sum_{cyc}x}+\frac{xyz\left(\sum_{cyc}\frac{1}{x}\right)^3}{6\sum_{cyc}x}\overset{?}{\ge}\frac{9}{\sum_{cyc}x}$$

$$\Leftrightarrow\frac{xyz\left(\sum_{cyc}\frac{1}{x}\right)^3}{6\sum_{cyc}x}\overset{?}{\ge}\frac{9}{2\sum_{cyc}x}\Leftrightarrow\left(\sum_{cyc}xy\right)^3\overset{?}{\ge}27x^2y^2z^2\to\text{true via AM}-\text{GM}$$

$$\therefore\text{LHS}\ge\frac{1}{3^{n-2}}.\left(\sum_{cyc}\frac{1}{x}\right)^{n-2}.\frac{9}{\sum_{cyc}x}\overset{\text{Bergstrom}}{\ge}\frac{1}{3^{n-2}}.\left(\frac{9}{\sum_{cyc}x}\right)^{n-2}.\frac{9}{\sum_{cyc}x}$$

$$=\frac{1}{3^{n-2}}.\left(\frac{9}{\sum_{cyc}x}\right)^{n-1}=\frac{1}{3^{n-2}}.\left(\frac{9}{h_a+w_b+m_c}\right)^{n-1}\ge\frac{1}{3^{n-2}}.\left(\frac{9}{\sum_{cyc}m_a}\right)^{n-1}$$

$$\overset{\substack{\text{Leuenberger}\\ and\\ \text{Euler}}}{\ge}\frac{1}{3^{n-2}}.\left(\frac{9}{\frac{9R}{2}}\right)^{n-1}$$

$$\therefore\frac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\frac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\frac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}\ge\frac{1}{3^{n-2}}\left(\frac{2}{R}\right)^{n-1}\quad\forall\, n\in\mathbb{N}-\{1,2,3\}$$

$\therefore$ combining *all cases*, $\dfrac{h_a^2+w_bm_c}{h_a^n(w_b+m_c)}+\dfrac{w_b^2+m_ch_a}{w_b^n(m_c+h_a)}+\dfrac{m_c^2+h_aw_b}{m_c^n(h_a+w_b)}$

$\ge\dfrac{1}{3^{n-2}}\left(\dfrac{2}{R}\right)^{n-1}$ $\forall\,\Delta$ ABC *and* $\forall\, n\in\mathbb{N}$," = " iff Δ ABC is equi*lateral* (QED)

1571. In any Δ ABC, the following relationship holds :

$$12 \le \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \le \frac{3R^2}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} = \sum_{cyc} \frac{(bc + ca)(bc + ab)}{ca.ab} = \sum_{cyc} \frac{(a + b)(c + a)}{a^2}$$

$$= \sum_{cyc} \frac{a^2 + \sum_{cyc} ab}{a^2} = 3 = 3 + \frac{1}{16R^2r^2s^2}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right)$$

$$\ge 3 + \frac{1}{48R^2r^2s^2}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} ab\right)^2 \overset{\text{via } (\bullet)}{\ge} 3 + \frac{24Rrs^2}{48R^2r^2s^2}(s^2 + 4Rr + r^2)$$

$$= \frac{s^2 + 10Rr + r^2}{2Rr} \overset{\text{Gerretsen}}{\ge} \frac{16Rr - 5r^2 + 10Rr + r^2}{2Rr} = \frac{26Rr - 4r^2}{2Rr} \overset{\text{Euler}}{\ge} \frac{26Rr - 2Rr}{2Rr}$$

$$= 12 \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \ge 12$$

$$\text{Again,} \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} = \sum_{cyc} \frac{(a + b)(c + a)}{a^2} \le \sum_{cyc} \frac{(a + b)(c + a)}{4(s - b)(s - c)}$$

$$= \frac{2s(s^2 + 2Rr + r^2)}{4r^2 s} . \sum_{cyc} \frac{s - a}{(b + c)} = \frac{s^2 + 2Rr + r^2}{2r^2} . \sum_{cyc} \frac{2s - a - s}{(b + c)}$$

$$= \frac{s^2 + 2Rr + r^2}{2r^2} . \left(3 - s \sum_{cyc} \frac{1}{b + c}\right) \overset{\text{Bergstrom}}{\le} \frac{s^2 + 2Rr + r^2}{2r^2} . \left(3 - \frac{9s}{4s}\right) \overset{\text{Gerretsen}}{\le}$$

$$\frac{4R^2 + 6Rr + 4r^2}{2r^2} . \frac{3}{4} \overset{\text{Euler}}{\le} \frac{4R^2 + 3R^2 + R^2}{2r^2} . \frac{3}{4} \therefore \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \le \frac{3R^2}{r^2}$$

$$\therefore 12 \le \sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \le \frac{3R^2}{r^2}$$

$\forall \Delta$ ABC, $'' ='' $ iff Δ ABC is equilateral (QED)

1572. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} \frac{(h_a + h_b)(h_a + h_c)}{h_b h_c} \ge \left(\frac{2r}{R}\right)^3 \sum_{cyc} \frac{(r_a + r_b)(r_a + r_c)}{r_b r_c}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} ab\right)^2 \overset{?}{\geq} 24Rrs^2 \Leftrightarrow \left(s^2+4Rr+r^2\right)^2 \overset{?}{\geq} 24Rrs^2$$

$$\Leftrightarrow s^4+2\left(4Rr+r^2\right)s^2+\left(4Rr+r^2\right)^2 \overset{?}{\geq} 24Rrs^2$$

$$\Leftrightarrow s^4-\left(16Rr-2r^2\right)s^2+\left(4Rr+r^2\right)^2 \underset{(1)}{\overset{?}{\geq}} 0$$

Now, LHS of (1) $\overset{\text{Gerretsen}}{\geq} \left(16Rr-5r^2\right)s^2-\left(16Rr-2r^2\right)s^2+\left(4Rr+r^2\right)^2$

$$= r^2\left((4R+r)^2-3s^2\right) \overset{\text{Trucht}}{\geq} 0 \Rightarrow (1) \text{ is true } \therefore \left(\sum_{cyc} ab\right)^2 \overset{(\bullet)}{\geq} 24Rrs^2$$

$$\text{Now, } \sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} = \sum_{cyc}\frac{(bc+ca)(bc+ab)}{ca.ab} = \sum_{cyc}\frac{(a+b)(c+a)}{a^2}$$

$$= \sum_{cyc}\frac{a^2+\sum_{cyc} ab}{a^2} = 3+\frac{1}{16R^2r^2s^2}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right)$$

$$\geq 3+\frac{1}{48R^2r^2s^2}\left(\sum_{cyc} ab\right)\left(\sum_{cyc} ab\right)^2 \overset{\text{via }(\bullet)}{\geq} 3+\frac{24Rrs^2}{48R^2r^2s^2}\left(s^2+4Rr+r^2\right)$$

$$= \frac{s^2+10Rr+r^2}{2Rr} \overset{\text{Gerretsen}}{\geq} \frac{16Rr-5r^2+10Rr+r^2}{2Rr}$$

$$\Rightarrow \sum_{cyc}\frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \overset{(\blacksquare)}{\geq} \frac{13R-2r}{R}$$

$$\text{Again, } \sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_b r_c} = \sum_{cyc}\frac{\left(\frac{rs}{s-a}+\frac{rs}{s-b}\right)\left(\frac{rs}{s-a}+\frac{rs}{s-c}\right)}{\frac{rs}{s-b}\cdot\frac{rs}{s-c}}$$

$$= \sum_{cyc}\frac{\frac{2s-a-b}{(s-a)(s-b)}\cdot\frac{2s-a-c}{(s-a)(s-c)}}{\frac{1}{(s-b)(s-c)}} = \sum_{cyc}\frac{bc}{(s-a)^2} \overset{A-G}{\leq} \sum_{cyc}\frac{(b+c)^2}{4(s-a)^2} = \sum_{cyc}\frac{(s+s-a)^2}{4(s-a)^2}$$

$$= \sum_{cyc}\frac{s^2+(s-a)^2+2s(s-a)}{4(s-a)^2} = \frac{3}{4}+\frac{1}{2r}\sum_{cyc}\frac{rs}{s-a}+\frac{1}{4r^2}\sum_{cyc}\frac{r^2s^2}{(s-a)^2}$$

$$= \frac{3}{4}+\frac{1}{2r}\sum_{cyc} r_a+\frac{1}{4r^2}\sum_{cyc} r_a^2 = \frac{3}{4}+\frac{4R+r}{2r}+\frac{(4R+r)^2-2s^2}{4r^2}$$

$$= \frac{(4R+r)^2-2s^2+2r(4R+r)+3r^2}{4r^2}$$

$$\overset{\text{Gerretsen}}{\leq} \frac{(4R+r)^2-2\left(16Rr-5r^2\right)+2r(4R+r)+3r^2}{4r^2}$$

$$\therefore \left(\frac{2r}{R}\right)^3 \sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{r_b r_c} \overset{(\blacksquare\blacksquare)}{\leq} \frac{32r(R^2-Rr+r^2)}{R^3} \therefore (\blacksquare),(\blacksquare\blacksquare) \Rightarrow$$

$$\text{in order to prove}: \sum_{cyc} \frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \geq \left(\frac{2r}{R}\right)^3 \sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{r_b r_c},$$

$$\text{it suffices to prove}: \frac{13R-2r}{R} \geq \frac{32r(R^2-Rr+r^2)}{R^3}$$

$$\Leftrightarrow 13t^3-34t^2+32t-32 \geq 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(9t^2+4t(t-2)+16) \geq 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \sum_{cyc} \frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \geq \left(\frac{2r}{R}\right)^3 \sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{r_b r_c}$$

$\forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC$ is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc} \frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} = \frac{(h_a+h_b)(h_b+h_c)(h_a+h_c)}{h_a h_b h_c} \cdot \sum_{cyc} \frac{h_a}{h_b+h_c} \underset{Nesbitt}{\overset{Cesaro}{\geq}} 8 \cdot \frac{3}{2} = 12.$$

$$\sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{r_b r_c} = \frac{4R}{r} \cdot \sum_{cyc} \frac{r_a}{r_b+r_c} \overset{CBS}{\leq} \frac{4R}{r} \cdot \sum_{cyc} \frac{r_a}{4}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)$$

$$= \frac{R}{r}\left(\sum_{cyc} r_a \cdot \sum_{cyc} \frac{1}{r_a} - 3\right) = \frac{R}{r}\left(\frac{4R+r}{r} - 3\right) =$$

$$= \frac{3R^3}{2r^3} - \frac{R}{r}\left(\frac{R}{2r}-1\right)\left(\frac{3R}{r}-2\right) \overset{Euler}{\leq} \frac{3R^3}{2r^3}.$$

Therefore

$$\sum_{cyc} \frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \geq 12 = \left(\frac{2r}{R}\right)^3 \cdot \frac{3R^3}{2r^3} \geq \left(\frac{2r}{R}\right)^3 \sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{r_b r_c}$$

Equality holds iff ΔABC is equilateral.

1573. In any $\Delta\, ABC$, the following relationship holds:

$$\frac{6R}{r} \leq \sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{r_b r_c} \leq \frac{3R^3}{2r^3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{(r_a+r_b)(r_a+r_c)}{r_b r_c} = \sum_{cyc} \frac{\left(\frac{rs}{s-a}+\frac{rs}{s-b}\right)\left(\frac{rs}{s-a}+\frac{rs}{s-c}\right)}{\frac{rs}{s-b} \cdot \frac{rs}{s-c}}$$

$$=\sum_{cyc}\frac{\frac{2s-a-b}{(s-a)(s-b)}\cdot\frac{2s-a-c}{(s-a)(s-c)}}{\frac{1}{(s-b)(s-c)}}=\sum_{cyc}\frac{bc}{(s-a)^2}\overset{A-G}{\le}\sum_{cyc}\frac{(b+c)^2}{4(s-a)^2}=\sum_{cyc}\frac{(s+s-a)^2}{4(s-a)^2}$$

$$=\sum_{cyc}\frac{s^2+(s-a)^2+2s(s-a)}{4(s-a)^2}=\frac{3}{4}+\frac{1}{2r}\sum_{cyc}\frac{rs}{s-a}+\frac{1}{4r^2}\sum_{cyc}\frac{r^2s^2}{(s-a)^2}$$

$$=\frac{3}{4}+\frac{1}{2r}\sum_{cyc}r_a+\frac{1}{4r^2}\sum_{cyc}r_a^2=\frac{3}{4}+\frac{4R+r}{2r}+\frac{(4R+r)^2-2s^2}{4r^2}$$

$$=\frac{(4R+r)^2-2s^2+2r(4R+r)+3r^2}{4r^2}$$

$$\overset{Gerretsen}{\le}\frac{(4R+r)^2-2(16Rr-5r^2)+2r(4R+r)+3r^2}{4r^2}=\frac{4(R^2-Rr+r^2)}{r^2}\overset{?}{\le}\frac{3R^3}{2r^3}$$

$$\Leftrightarrow 3t^3-8t^2+8t-8\ge 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow (t-2)\big(2t^2+t(t-2)+4\big)\ge 0\to \text{true}$$

$$\because t\overset{Euler}{\ge}2\ \therefore \sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\le\frac{3R^3}{2r^3}$$

$$\text{Again},\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\sum_{cyc}\frac{bc}{(s-a)^2}=\frac{1}{r}\sum_{cyc}\left(\frac{bc}{s(s-a)}\cdot\frac{rs}{s-a}\right)$$

$$=\frac{1}{r}\sum_{cyc}\left(\sec^2\frac{A}{2}\cdot r_a\right)\overset{Chebyshev}{\ge}\frac{1}{3r}\left(\sum_{cyc}\sec^2\frac{A}{2}\right)\left(\sum_{cyc}r_a\right)$$

$$\left(\because \text{WLOG assuming } a\ge b\ge c\Rightarrow \sec^2\frac{A}{2}\ge\sec^2\frac{B}{2}\ge\sec^2\frac{C}{2}\text{ and } r_a\ge r_b\ge r_c\right)$$

$$=\frac{1}{3r}\cdot\frac{s^2+(4R+r)^2}{s^2}\cdot(4R+r)\overset{?}{\ge}\frac{6R}{r}\Leftrightarrow (4R+r)^3+s^2(4R+r)\overset{?}{\ge}18Rs^2$$

$$\Leftrightarrow (4R+r)^3\underset{(*)}{\overset{?}{\ge}}(14R-r)s^2,\text{ but }(14R-r)s^2\overset{Gerretsen}{\le}$$

$$(14R-r)(4R^2+4Rr+3r^2)\overset{?}{\le}(4R+r)^3\Leftrightarrow 4t^3-2t^2-13t+2\overset{?}{\ge}0$$

$$\Leftrightarrow (t-2)(4t^2+6t-1)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2\ \therefore\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\ge\frac{6R}{r}$$

$$\text{and hence},\frac{6R}{r}\le\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}\le\frac{3R^3}{2r^3}\ \forall\,\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{(r_a+r_b)(r_b+r_c)(r_a+r_c)}{r_ar_br_c}\cdot\sum_{cyc}\frac{r_a}{r_b+r_c}$$

$$=\frac{4Rs^2}{s^2r}\cdot\sum_{cyc}\frac{r_a}{r_b+r_c}\overset{Nesbitt}{\ge}\frac{6R}{r}.$$

$$\sum_{cyc}\frac{(r_a+r_b)(r_a+r_c)}{r_br_c}=\frac{4R}{r}\cdot\sum_{cyc}\frac{r_a}{r_b+r_c}\overset{CBS}{\leq}\frac{4R}{r}\cdot\sum_{cyc}\frac{r_a}{4}\left(\frac{1}{r_b}+\frac{1}{r_c}\right)$$

$$=\frac{R}{r}\left(\sum_{cyc}r_a\cdot\sum_{cyc}\frac{1}{r_a}-3\right)$$

$$=\frac{R}{r}\left(\frac{4R+r}{r}-3\right)=\frac{3R^3}{2r^3}-\frac{R}{r}\left(\frac{R}{2r}-1\right)\left(\frac{3R}{r}-2\right)\overset{Euler}{\leq}\frac{3R^3}{2r^3}.$$

which completes the proof. Equality holds iff ΔABC is equilateral.

1574. In ΔABC the following relationship holds:

$$\frac{1-\cos(B-C)}{h_a}+\frac{1-\cos(C-A)}{h_b}+\frac{1-\cos(A-B)}{h_c}=\frac{R-2r}{Rr}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{1-\cos(B-C)}{h_a}=\sum_{cyc}\frac{1}{h_a}-\sum_{cyc}\frac{\cos(B-C)}{h_a}=$$

$$=\sum_{cyc}\frac{1}{\frac{2F}{a}}-\sum_{cyc}\frac{\cos(B-C)}{\frac{2F}{a}}=\frac{1}{2F}\sum_{cyc}a-\frac{1}{2F}\sum_{cyc}a\cos(B-C)=$$

$$=\frac{2s}{2F}-\frac{1}{2F}\sum_{cyc}2R\sin A\cos(B-C)=\frac{s}{F}-\frac{R}{F}\sum_{cyc}\sin A\cos(B-C)=$$

$$=\frac{s}{rs}-\frac{R}{2F}\sum_{cyc}(\sin(A+B-C)+\sin(A-B+C))=$$

$$=\frac{1}{r}-\frac{R}{2F}\sum_{cyc}\left(\sin(\pi-2C)+\sin(\pi-2B)\right)=$$

$$=\frac{1}{r}-\frac{R}{2F}\sum_{cyc}(\sin 2C+\sin 2B)=\frac{1}{r}-\frac{R}{F}\sum_{cyc}\sin 2A=$$

$$=\frac{1}{r}-\frac{R}{F}\cdot 4\sin A\sin B\sin C=\frac{1}{r}-\frac{R}{F}\cdot 4\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}=$$

$$=\frac{1}{r}-\frac{abc}{F\cdot 2R^2}=\frac{1}{r}-\frac{4RF}{F\cdot 2R^2}=\frac{1}{r}-\frac{2}{R}=\frac{R-2r}{Rr}$$

1575. In any Δ ABC, the following relationship holds :

$$\frac{r_b+r_c}{2r_a+n_a}+\frac{r_c+r_a}{2r_b+n_b}+\frac{r_a+r_b}{2r_c+n_c}\ge 2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_b+r_c}{2r_a+n_a}+\frac{r_c+r_a}{2r_b+n_b}+\frac{r_a+r_b}{2r_c+n_c}=\sum_{cyc}\frac{(r_b+r_c)^2}{2r_a(r_b+r_c)+n_a(r_b+r_c)}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}(r_b+r_c)\right)^2}{2\sum_{cyc}r_a(r_b+r_c)+\sum_{cyc}n_a(r_b+r_c)}=\frac{4(4R+r)^2}{4s^2+\sum_{cyc}n_a(r_b+r_c)}\overset{?}{\ge}2$$

$$\Leftrightarrow \boxed{2(4R+r)^2-4s^2\underset{(*)}{\overset{?}{\ge}}\sum_{cyc}n_a(r_b+r_c)}$$

$$\text{Now, } r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Also, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2$$

$$\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}$$

$$=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)$$

$$=as^2-2ah_ar_a\Rightarrow n_a^2=s^2-2h_ar_a=s^2-\frac{4rs^2\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}}=s^2\left(1-\frac{r}{R}\sec^2\frac{A}{2}\right)$$

$$\Rightarrow n_a(r_b+r_c)\overset{\text{via (i)}}{=}4Rs.\sqrt{1-\frac{r}{R}\sec^2\frac{A}{2}}.\cos^2\frac{A}{2}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}n_a(r_b+r_c)=4Rs.\sum_{cyc}\left(\left(\sqrt{1-\frac{r}{R}\sec^2\frac{A}{2}}.\cos\frac{A}{2}\right)\left(\cos\frac{A}{2}\right)\right)\overset{CBS}{\le}$$

$$4Rs.\sqrt{\sum_{cyc}\left(\left(1-\frac{r}{R}\sec^2\frac{A}{2}\right)\cos^2\frac{A}{2}\right)}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}}$$

$$=4Rs.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}-\frac{3r}{R}}.\sqrt{\sum_{cyc}\cos^2\frac{A}{2}}=4Rs.\sqrt{\left(\frac{4R+r}{2R}-\frac{3r}{R}\right)\left(\frac{4R+r}{2R}\right)}$$

$$= 2s.\sqrt{(4R+r)(4R-5r)} \overset{?}{\leq} 2(4R+r)^2 - 4s^2$$

$$\Leftrightarrow (4R+r)^4 + 4s^4 - 4s^2(4R+r)^2 \overset{?}{\geq} s^2(4R+r)(4R-5r)$$

$$\Leftrightarrow \boxed{4s^4 + (4R+r)^4 - s^2(4R+r)(20R-r) \underset{(**)}{\overset{?}{\geq}} 0} \text{ and}$$

$$\because 4(s^2 - 4R^2 - 4Rr - 3r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$4s^4 + (4R+r)^4 - s^2(4R+r)(20R-r) \geq 4(s^2 - 4R^2 - 4Rr - 3r^2)^2$$

$$\Leftrightarrow \boxed{(48R^2 - 16Rr - 25r^2)s^2 \overset{(***)}{\leq} 192R^4 + 128R^3r - 64R^2r^2 - 80Rr^3 - 35r^4}$$

$$\text{Now, LHS of } (***) \overset{\text{Rouche}}{\leq} (48R^2 - 16Rr - 25r^2)\left(\begin{matrix} 2R^2 + 10Rr - r^2 \\ +2(R-2r).\sqrt{R^2 - 2Rr} \end{matrix}\right)$$

$$\overset{?}{\leq} 192R^4 + 128R^3r - 64R^2r^2 - 80Rr^3 - 35r^4$$

$$\Leftrightarrow \boxed{(R-2r)(48R^3 - 64R^2r - 31Rr^2 + 15r^3) \underset{(****)}{\overset{?}{\geq}} (R-2r)(48R^2 - 16Rr - 25r^2).\sqrt{R^2 - 2Rr}}$$

$$\because 48R^3 - 64R^2r - 31Rr^2 + 15r^3 = (R-2r)(48R^2 + 32Rr + 33r^2) + 81r^3$$

$$\overset{\text{Euler}}{\geq} 81r^3 > 0 \text{ and } \because R - 2r \overset{\text{Euler}}{\geq} 0 \therefore \text{in order to prove } (****), \text{it suffices to prove :}$$

$$(48R^3 - 64R^2r - 31Rr^2 + 15r^3)^2 > (R^2 - 2Rr)(48R^2 - 16Rr - 25r^2)^2$$

$$\Leftrightarrow r^2(192R^4 + 320R^3r + 16R^2r^2 + 320Rr^3 + 225r^4) > 0 \to \text{true}$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{is true}$$

$$\therefore \frac{r_b + r_c}{2r_a + n_a} + \frac{r_c + r_a}{2r_b + n_b} + \frac{r_a + r_b}{2r_c + n_c} \geq 2$$

$$\forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1576. In any Δ ABC, the following relationship holds :

$$\frac{r_b + r_c}{r_a + 2m_a} + \frac{r_c + r_a}{r_b + 2m_b} + \frac{r_a + r_b}{r_c + 2m_c} \geq 2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_b + r_c}{r_a + 2m_a} + \frac{r_c + r_a}{r_b + 2m_b} + \frac{r_a + r_b}{r_c + 2m_c} = \sum_{\text{cyc}} \frac{(r_b + r_c)^2}{r_a(r_b + r_c) + 2m_a(r_b + r_c)}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\text{cyc}}(r_b + r_c)\right)^2}{\sum_{\text{cyc}} r_a(r_b + r_c) + 2\sum_{\text{cyc}} m_a(r_b + r_c)} = \frac{4(4R+r)^2}{2s^2 + 2\sum_{\text{cyc}} m_a(r_b + r_c)} \overset{?}{\geq} 2$$

$$\Leftrightarrow (4R+r)^2 - s^2 \underset{(*)}{\overset{?}{\geq}} \sum_{\text{cyc}} m_a(r_b + r_c)$$

$$\text{Now, } \sum_{cyc} m_a(r_b + r_c) \overset{CBS}{\le} \sqrt{\sum_{cyc} m_a^2} \cdot \sqrt{\sum_{cyc}(r_b + r_c)^2}$$

$$= \sqrt{\frac{3}{4} \cdot 2(s^2 - 4Rr - r^2)} \cdot \sqrt{2\left(\left((4R + r)^2 - 2s^2\right) + s^2\right)}$$

$$= \sqrt{3(s^2 - 4Rr - r^2)((4R + r)^2 - s^2)} \overset{?}{\le} (4R + r)^2 - s^2$$

$$\Leftrightarrow 3(s^2 - 4Rr - r^2) \overset{?}{\le} (4R + r)^2 - s^2 \Leftrightarrow s^2 \overset{?}{\le} 4R^2 + 5Rr + r^2$$

$$\Leftrightarrow \left(s^2 - (4R^2 + 4Rr + 3r^2)\right) - r(R - 2r) \overset{?}{\le} 0 \to \text{true} \because s^2 - (4R^2 + 4Rr + 3r^2)$$

$$\overset{Gerretsen}{\le} 0 \text{ and } -r(R - 2r) \overset{Euler}{\le} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{r_b + r_c}{r_a + 2m_a} + \frac{r_c + r_a}{r_b + 2m_b} + \frac{r_a + r_b}{r_c + 2m_c} \ge 2$$

$$\forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1577.

Prove that in *all* triangle ABC holds :

$$\sum_{cyc}(1 - \cos A - \cos 2A - \cos(B - C))^2 = \left(\frac{s^2 - 4Rr - r^2}{R^2}\right)^2 - \left(\frac{s^2 + 4Rr + r^2}{2R^2}\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}(1 - \cos A - \cos 2A - \cos(B - C))^2$$

$$= \sum_{cyc}\left(1 - \cos 2A + (\cos(B + C) - \cos(B - C))\right)^2 = \sum_{cyc}\left(2\sin^2 A - 2\sin B \sin C\right)^2$$

$$= \sum_{cyc}\left(\frac{2a^2}{4R^2} - \frac{2bc}{4R^2}\right)^2 = \frac{1}{4R^4}\sum_{cyc}(a^4 + b^2c^2 - 2a^2bc)$$

$$= \frac{1}{4R^4}\left(\left(\sum_{cyc} a^4 + 2\sum_{cyc} b^2c^2\right) - \left(\sum_{cyc} b^2c^2 + 2abc\sum_{cyc} a\right)\right)$$

$$= \frac{1}{4R^4}\left(\left(\sum_{cyc} a^2\right)^2 - \left(\sum_{cyc} ab\right)^2\right) = \frac{4(s^2 - 4Rr - r^2)^2}{4R^4} - \frac{(s^2 + 4Rr + r^2)^2}{4R^4}$$

$$= \left(\frac{s^2 - 4Rr - r^2}{R^2}\right)^2 - \left(\frac{s^2 + 4Rr + r^2}{2R^2}\right)^2 \text{ (QED)}$$

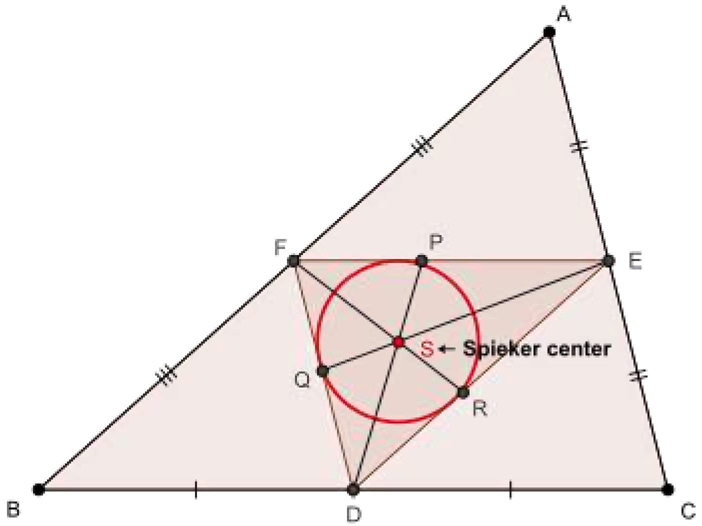
1578.

In any Δ ABC with p_a, p_b, p_c $\to$ Spieker cevians, the following relationship holds :

$$\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\le\frac{4R+r}{3r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow \mathbf{c}\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, \mathbf{b}\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}=$$

$$\binom{2s}{+a}.\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}(a=y+z, b=z+x, c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$\Rightarrow \boxed{ap_a^2 \overset{(\bullet\bullet\bullet)}{=} as(s-a)-\frac{a(b-c)^2}{4}+\frac{a(4s+a)^2}{(4s+2a)^2}.(b-c)^2}$$

$$\text{Now, }\frac{a(4s+a)^2}{(4s+2a)^2}=a.\frac{(4s+2a)^2-2a(4s+2a)+a^2}{(4s+2a)^2}$$

$$=a-\frac{(a+2s-2s)^2}{2s+a}+\frac{(a+2s-2s)^3}{(4s+2a)^2}$$

$$=a-(2s+a)+4s-\frac{4s^2}{2s+a}+\frac{1}{4}\left(\frac{(2s+a)^3-8s^3-3(2s+a)(2s)a}{(2s+a)^2}\right)$$

$$=2s-\frac{4s^2}{2s+a}+\frac{2s+a}{4}-\frac{2s^3}{(2s+a)^2}-\frac{3s(a+2s-2s)}{2(2s+a)}$$

$$=\frac{5s}{2}+\frac{a}{4}-\frac{4s^2}{2s+a}-\frac{2s^3}{(2s+a)^2}-\frac{3s}{2}+\frac{3s^2}{2s+a}$$

$$\therefore \boxed{\frac{a(4s+a)^2}{(4s+2a)^2}\overset{(\bullet\bullet\bullet\bullet)}{=}s+\frac{a}{4}-\frac{s^2(4s+a)}{(2s+a)^2}}$$

$$\therefore (\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet)\Rightarrow ap_a^2=$$

$$as(s-a)-\frac{a(b-c)^2}{4}+s(b-c)^2+\frac{a(b-c)^2}{4}$$

$$-\frac{s^2(4s+a)}{(2s+a)^2}.(b-c)^2\overset{a<s}{\le}as(s-a)+s(b-c)^2-\frac{s^2(4s+a)}{(2s+s)^2}.(b-c)^2$$

$$=as(s-a)+s(b-c)^2-\frac{(4s+a)(b-c)^2}{9}$$

$$=as(s-a)+\frac{5s(b-c)^2}{9}-\frac{a(b-c)^2}{9}\text{ and analogs}$$

$$\therefore \sum_{cyc}ap_a^2=s\left(2s-2(s^2-4Rr-r^2)\right)+\frac{10s}{9}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$-\frac{1}{9}\left(\sum_{cyc}(ab(2s-c))-6abc\right)$$

$$=s(8Rr+2r^2)+\frac{10s(s^2-12Rr-3r^2)}{9}-\frac{2s(s^2-14Rr+r^2)}{9}$$

$$=\frac{2s(4s^2-10Rr-7r^2)}{9}\Rightarrow\frac{\sum_{cyc}ap_a^2}{2rs}\overset{(\blacklozenge)}{\le}\frac{4s^2-10Rr-7r^2}{9r}\text{ and }\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\overset{CBS}{\le}$$

$$\sqrt{\sum_{cyc}\frac{p_a^2}{h_a}}.\sqrt{\sum_{cyc}\frac{1}{h_a}}=\sqrt{\frac{\sum_{cyc}ap_a^2}{2rs}}.\sqrt{\frac{1}{r}}\overset{\text{via }(\blacklozenge)}{\le}\sqrt{\frac{4s^2-10Rr-7r^2}{9r^2}}$$

$$\overset{\text{Gerretsen}}{\le}\sqrt{\frac{4(4R^2+4Rr+3r^2)-10Rr-7r^2}{9r^2}}$$

$$=\sqrt{\frac{16R^2+6Rr+5r^2}{9r^2}}\overset{\text{Euler}}{\le}\sqrt{\frac{16R^2+6Rr+r^2+2Rr}{9r^2}}=\sqrt{\frac{(4R+r)^2}{9r^2}}$$

$$\therefore\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\le\frac{4R+r}{3r}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1579. In any Δ ABC, the following relationship holds :

$$\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}\le\frac{3R}{2r}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}$$

$$=\frac{(4R+r)^2}{s^2}+\frac{6(4R+r)^2}{s\left(2s+2\sum_{cyc}\sqrt{ab}\right)}-\frac{(4R+r)^2}{s^2}\overset{GM-HM}{\le}\frac{6(4R+r)^2}{s\left(2s+4\sum_{cyc}\frac{ab}{a+b}\right)}$$

$$=\frac{3(4R+r)^2}{s\left(s+2.4Rrs.\sum_{cyc}\frac{1}{ca+bc}\right)}\overset{Bergstrom}{\le}\frac{3(4R+r)^2}{s\left(s+2.4Rrs..\frac{9}{2\sum_{cyc}ab}\right)}=\frac{3(4R+r)^2}{s^2+\frac{36Rrs^2}{s^2+4Rr+r^2}}\overset{?}{\le}\frac{3R}{2r}$$

$$\Leftrightarrow Rs^4+rs^2(8R^2-15Rr-2r^2)\underset{(*)}{\overset{?}{\ge}}2r^2(4R+r)^3$$

$$\text{Now, LHS of }(*)\overset{Gerretsen}{\ge}\left(R(16Rr-5r^2)+r(8R^2-15Rr-2r^2)\right)s^2$$

$$\overset{Gerretsen}{\ge}r(24R^2-20Rr-2r^2)(16Rr-5r^2)\overset{?}{\ge}2r^2(4R+r)^3$$

$$\Leftrightarrow 64t^3-134t^2+11t+2\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)\left(60t^2+3t(t-2)+(t^2-4)+3\right)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}\le\frac{3R}{2r}\ \forall\Delta ABC,$$

$$''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $r_a+r_b+r_c=4R+r$ and $r_ar_b+r_br_c+r_cr_a=s^2$, then we have

$$LHS=\frac{(4R+r)^2}{s^2}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}=\frac{6s(4R+r)^2}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}$$

$$\overset{Euler}{\le}\frac{4.6s\left(4R+\frac{R}{2}\right)^2}{(a+b+c)^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}\overset{AM-GM}{\le}\frac{486sR^2}{\left(3\sqrt[3]{abc}\right)^2\left(3\sqrt[6]{abc}\right)^2}=\frac{6sR^2}{abc}=\frac{3R}{2r},$$

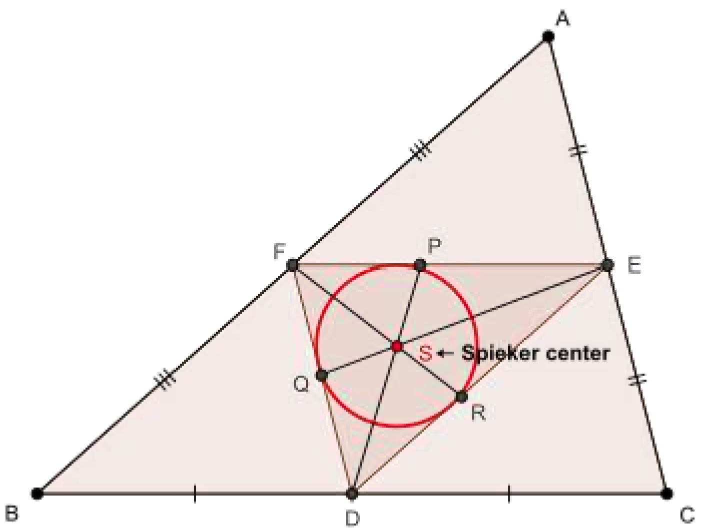
as desired. Equality holds iff ΔABC is equilateral.

1580. In any $\Delta\,ABC$ with $p_a, p_b, p_c \to$ Spieker cevians, the following relationship holds :

$$\frac{p_a p_b p_c}{r_a r_b r_c} \le \frac{8R - 7r}{9r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of $\Delta\,DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\,DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on $\Delta\,AFS$ and $\Delta\,AES$, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$
$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$
$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$
$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{We have}: \prod_{cyc}(2s+a) = 8s^3 + 4s^2\sum_{cyc} a + 2s\sum_{cyc} ab + 4Rrs$$

$$= 8s^3 + 4s^2.2s + 2s(s^2+4Rr+r^2) + 4Rrs$$

$$\Rightarrow \prod_{cyc}(2s+a) \overset{(\blacksquare\blacksquare)}{=} 2s(9s^2+6Rr+r^2)$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$= \sum_{cyc} a^3 - abc + 2a.2bc\cos A = 2s(s^2-6Rr-3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2) = 2s(Q+8Rr\cos A) \text{ and analogs}$$

$$(Q = s^2-8Rr-3r^2) \Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$= 8s^3\left(Q^3 + Q^2.8Rr\sum_{cyc}\cos A + Q.64R^2r^2.\sum_{cyc}\cos B\cos C + 512R^3r^3\prod_{cyc}\cos A\right)$$

$$= 8s^3\left(\begin{array}{c} Q^3 + Q^2.8Rr.\frac{R+r}{R} + Q.32R^2r^2.\left(\left(\frac{R+r}{R}\right)^2 - \left(3 - \frac{2(s^2-4Rr-r^2)}{4R^2}\right)\right) \\ +512R^3r^3.\frac{s^2-(2R+r)^2}{4R^2} \end{array}\right)$$

$$\Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a(4m_a^2)\right) \overset{(\blacksquare\blacksquare\blacksquare)}{=}$$

$$8s^3\left(\begin{array}{c} (s^2-8Rr-3r^2)^3 + (s^2-8Rr-3r^2)^2.8r(R+r) + \\ (s^2-8Rr-3r^2).16r^2(s^2-4R^2+r^2) + 128Rr^3(s^2-(2R+r)^2) \end{array}\right)$$

$$\therefore (\blacksquare), (\blacksquare\blacksquare), (\blacksquare\blacksquare\blacksquare)$$

$$\Rightarrow \boxed{\prod_{cyc}\frac{p_a^2}{r_a^2} = \frac{8s^3.8s^3\left(\begin{matrix}(s^2-8Rr-3r^2)^3+(s^2-8Rr-3r^2)^2.8r(R+r)+\\(s^2-8Rr-3r^2).16r^2(s^2-4R^2+r^2)+128Rr^3(s^2-(2R+r)^2)\end{matrix}\right)}{4s^2(9s^2+6Rr+r^2)^2.r^2s^4}}$$

$$\overset{?}{\leq}\left(\frac{8R-7r}{9r}\right)^2$$

$$\Leftrightarrow \boxed{\begin{matrix}1296s^6-(5184R^2+11664Rr-15471r^2)s^4-rs^2\left(\begin{matrix}6912R^3-10944R^2r+\\44748Rr^2+69570r^3\end{matrix}\right)\\-r^2(2304R^4-3264R^3r+484R^2r^2+21212Rr^3+3937r^4)\underset{(\bullet)}{\overset{?}{\leq}}0\end{matrix}}$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\geq 0 \text{ and } s^2-(m+n)\leq 0, \text{where}$$

$$m = 2R^2+10Rr-r^2 \text{ and } n = 2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0 \Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(\blacklozenge)}{\leq} 0$$

$$\therefore 1296s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\leq 0 \Rightarrow \text{in order to prove } (\bullet),$$

$$\text{it suffices to prove : LHS of } (\bullet)$$

$$\leq 1296s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)$$

$$\Leftrightarrow \boxed{\begin{matrix}(14256R+12879r)s^4-rs^2(51264R^2+60300Rr+70866r^2)\\-r(2304R^4-3264R^3r+484R^2r^2+21212Rr^3+3937r^4)\underset{(\bullet\bullet)}{\overset{?}{\leq}}0\end{matrix}} \text{ and}$$

$$\because (14256R+12879r)(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3) \overset{\text{via }(\blacklozenge)}{\leq} 0$$

$$\therefore \text{in order to prove } (\bullet), \text{it suffices to prove :}$$

$$\text{LHS of } (\bullet\bullet) \leq (14256R+12879r)(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)$$

$$\Leftrightarrow \boxed{\begin{matrix}(8208R^3-71343R^2r-42192Rr^2+24156r^3)s^2\\+r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4) \overset{(\bullet\bullet\bullet)}{\geq} 0\end{matrix}}$$

Case 1 $8208R^3-71343R^2r-42192Rr^2+24156r^3\geq 0$ *and* then : LHS of $(\bullet\bullet\bullet)$

$$\geq r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4)>0$$

$$\Rightarrow (\bullet\bullet\bullet) \text{ is true}$$

Case 2 $8208R^3-71343R^2r-42192Rr^2+24156r^3<0$ *and* then : LHS of $(\bullet\bullet\bullet)$

$$= -\left(-(8208R^3-71343R^2r-42192Rr^2+24156r^3)\right)s^2$$

$$+r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4)$$

$$\overset{\text{Gerretsen}}{\geq} -\left(-(8208R^3-71343R^2r-42192Rr^2+24156r^3)\right)(4R^2+4Rr+3r^2)$$

$$+r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow 8208t^5-5967t^4-13299t^3-22184t^2+4388t+19168\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(8208t^3+26865t^2+61329t+115672)+221760\right)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(\bullet\bullet\bullet)\text{ is true }\therefore\text{ combining both cases,}$$

$$\forall\Delta ABC,(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}\Rightarrow\forall\Delta ABC,\prod_{cyc}\frac{p_a^2}{r_a^2}\leq\left(\frac{8R-7r}{9r}\right)^2$$

$$\Rightarrow\frac{p_ap_bp_c}{r_ar_br_c}\leq\frac{8R-7r}{9r}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

1581. In any Δ ABC, the following relationship holds :

$$\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}\geq 5$$

Proposed by Adil Abdullayev-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Tereshin}\Rightarrow m_a\geq\frac{b^2+c^2}{4R}\Rightarrow\frac{4RFm_a}{F}\geq b^2+c^2\Rightarrow\frac{abcm_a}{F}\geq b^2+c^2$$

$$\Rightarrow\frac{am_a}{F}\geq\frac{b}{c}+\frac{c}{b}\text{ implementing which on a triangle with sides }\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$$

$$\text{whose area via elementary calculations}=\frac{F}{3}\text{ and medians}=\frac{a}{2},\frac{b}{2},\frac{c}{2},$$

$$\text{we get}:\frac{\left(\frac{2m_a}{3}\right)\left(\frac{a}{2}\right)}{\frac{F}{3}}\geq\frac{\frac{2m_b}{3}}{\frac{2m_c}{3}}+\frac{\frac{2m_c}{3}}{\frac{2m_b}{3}}\Rightarrow\frac{2m_a}{\left(\frac{2F}{a}\right)}\geq\frac{m_b}{m_c}+\frac{m_c}{m_b}\Rightarrow\frac{2m_a}{h_a}\geq\frac{m_b}{m_c}+\frac{m_c}{m_b}\rightarrow(1)$$

$$\therefore\text{ via (1) and analogs, }\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}\geq\sum_{cyc}\sqrt{\frac{1}{2}\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)}$$

$$=\sum_{cyc}\frac{m_b^2+m_c^2}{\sqrt{2m_bm_c(m_b^2+m_c^2)}}\geq\sum_{cyc}\frac{2(m_b^2+m_c^2)}{\sqrt{8m_bm_c(m_b^2+m_c^2)}}=\sum_{cyc}\frac{2(m_b^2+m_c^2)}{(m_b+m_c)^2}$$

$$\left(\because(y+z)^4=(y^2+z^2+2yz)^2\overset{A-G}{\geq}8yz(y^2+z^2)\right)$$

$$\Rightarrow\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}$$

$$\geq\sum_{cyc}\frac{2(m_b^2+m_c^2)}{(m_b+m_c)^2}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}\overset{?}{\geq}5$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{2(y^2+z^2)}{(y+z)^2}-1\right) \overset{?}{\geq} 2-\frac{6\sum_{cyc}xy}{\left(\sum_{cyc}x\right)^2}\ (x=m_a, y=m_b, z=m_c)$$

$$\Leftrightarrow \boxed{\sum_{cyc}\frac{(y-z)^2}{(y+z)^2} \underset{(*)}{\overset{?}{\geq}} \frac{2\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)}{\left(\sum_{cyc}x\right)^2}}$$

Assigning $y+z=X, z+x=Y, x+y=Z \Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0 \Rightarrow X+Y>Z, Y+Z>X, Z+X>Y \Rightarrow X, Y, Z$ form sides of triangle with semiperimeter, circumradius and inradius $= s', R', r'$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s' \Rightarrow \sum_{cyc}x=s' \to (1) \Rightarrow x=s'-X, y=s'-Y,$ $z=s'-Z$ and suchsubstitutions $\Rightarrow xyz=r'^2s' \to (2)$,

$$\sum_{cyc}xy=\sum_{cyc}(s'-X)(s'-Y) \Rightarrow \sum_{cyc}xy=4R'r'+r'^2 \to (3) \text{ and}$$

$$\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy \overset{\text{via (1) and (3)}}{=} s'^2-2(4R'r'+r'^2)$$

$$\Rightarrow \sum_{cyc}x^2=s'^2-8R'r'-2r'^2 \to (4) \therefore (*)$$

$$\Leftrightarrow \sum_{cyc}\frac{\left((s'-Y)-(s'-Z)\right)^2}{X^2} \geq \frac{2\left(s'^2-8R'r'-2r'^2-(4R'r'+r'^2)\right)}{s'^2}$$

$$\Leftrightarrow \boxed{\sum_{cyc}\frac{(Y-Z)^2}{X^2} \overset{(**)}{\geq} \frac{2(s'^2-12R'r'-3r'^2)}{s'^2}}$$

$$\text{Now, } \sum_{cyc}b^2c^2(b-c)^2=\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-3a^2b^2c^2-2\sum_{cyc}a^3b^3$$

$$=\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-3a^2b^2c^2$$

$$-2\left(3a^2b^2c^2+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2-abc\sum_{cyc}a\right)\right)$$

$$=2\left((s^2+4Rr+r^2)^2-16Rrs^2\right)\left((s^2-4Rr-r^2)-(s^2+4Rr+r^2)\right)-144R^2r^2s^2$$
$$+2.4Rrs.2s(s^2+4Rr+r^2)=4r^2\left((12R^2+4Rr-2r^2)s^2-s^4-r(4R+r)^3\right)$$

$$\Rightarrow \sum_{cyc}\frac{(b-c)^2}{a^2}=\frac{\sum_{cyc}b^2c^2(b-c)^2}{16R^2r^2s^2}$$

$$=\frac{4r^2\left((12R^2+4Rr-2r^2)s^2-s^4-r(4R+r)^3\right)}{16R^2r^2s^2} \overset{?}{\geq} \frac{2(s^2-12Rr-3r^2)}{s^2}$$

$$\Leftrightarrow \boxed{s^4-(4R^2+4Rr-2r^2)s^2-r(32R^3-24R^2r-12Rr^2-r^3)\underset{(***)}{\overset{?}{\le}}0}$$

$$\text{Now, LHS of }(***)\overset{\text{Gerretsen}}{\le}\left((4R^2+4Rr+3r^2)-(4R^2+4Rr-2r^2)\right)s^2$$

$$-r(32R^3-24R^2r-12Rr^2-r^3)=5r^2s^2-r(32R^3-24R^2r-12Rr^2-r^3)$$

$$\overset{\text{Gerretsen}}{\le}5r^2(4R^2+4Rr+3r^2)-r(32R^3-24R^2r-12Rr^2-r^3)\overset{?}{\le}0$$

$$\Leftrightarrow 8t^3-11t^2-8t-4\overset{?}{\ge}0\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(8t^2+5t+2)\overset{?}{\ge}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\ge}2\Rightarrow(***)\text{ is true}\therefore\sum_{\text{cyc}}\frac{(b-c)^2}{a^2}\ge\frac{2(s^2-12Rr-3r^2)}{s^2}$$

$$\Rightarrow\sum_{\text{cyc}}\frac{(Y-Z)^2}{X^2}\overset{(**)}{\ge}\frac{2(s'^2-12R'r'-3r'^2)}{s'^2}\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\Rightarrow\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}\ge 5$$

$\forall\,\Delta\,ABC$," = " iff $\Delta\,ABC$ is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Tereshin's inequality, we have

$$\frac{m_a}{h_a}\ge\frac{b^2+c^2}{4Rh_a}=\frac{b^2+c^2}{2bc}.$$

Applying this inequality to the triangle GBC
(G is the centroid of the triangle ABC) and noting that
the altitude from G is equal to $\frac{h_a}{3}$ and $GB=\frac{2}{3}m_b$, $GC=\frac{2}{3}m_c$, we get

$$\frac{\frac{m_a}{3}}{\frac{h_a}{3}}\ge\frac{\left(\frac{2}{3}m_b\right)^2+\left(\frac{2}{3}m_c\right)^2}{2.\frac{2}{3}m_b.\frac{2}{3}m_c}\quad\text{or}\quad\frac{m_a}{h_a}\ge\frac{{m_b}^2+{m_c}^2}{2m_bm_c}.$$

$$\Rightarrow\sqrt{\frac{m_a}{h_a}}\ge\sqrt{\frac{{m_b}^2+{m_c}^2}{2m_bm_c}}=\frac{2({m_b}^2+{m_c}^2)}{2\sqrt{2m_bm_c({m_b}^2+{m_c}^2)}}\overset{AM-GM}{\ge}\frac{2({m_b}^2+{m_c}^2)}{2m_bm_c+({m_b}^2+{m_c}^2)}$$

$$=\frac{(m_b-m_c)^2}{(m_b+m_c)^2}+1\ge\frac{(m_b-m_c)^2}{(m_a+m_b+m_c)^2}+1\ \text{(and analogs)}$$

Therefore

$$\sqrt{\frac{m_a}{h_a}}+\sqrt{\frac{m_b}{h_b}}+\sqrt{\frac{m_c}{h_c}}+\frac{6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}\ge$$

$$\ge\frac{(m_b-m_c)^2+(m_c-m_a)^2+(m_a-m_b)^2+6(m_am_b+m_bm_c+m_cm_a)}{(m_a+m_b+m_c)^2}+3=5,$$

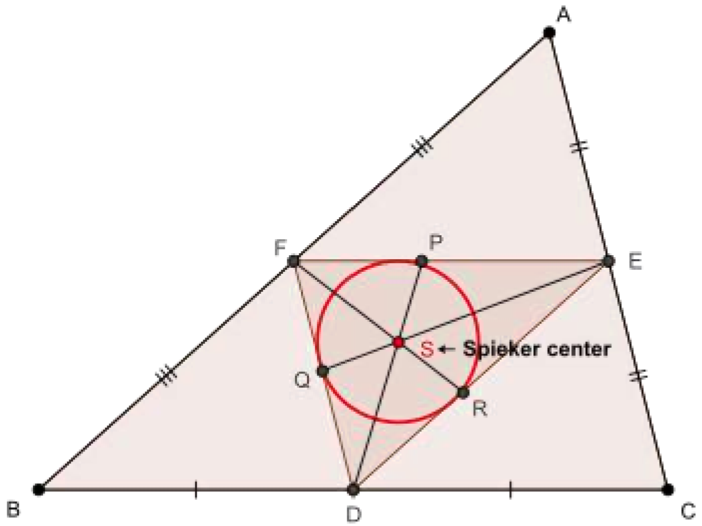
as desired. Equality holds iff ΔABC is equilateral.

1582. If p_a, p_b, p_c are the Spieker's cevians in ΔABC then:

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}\geq\frac{2}{R}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$(say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow(1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\text{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(\text{ii})}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

Now, $b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$

$$= \sum_{\text{cyc}} a^3 - abc + 2a.2bc\cos A = 2s(s^2-6Rr-3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2) \overset{(\blacksquare\blacksquare)}{=} 2s(s^2-8Rr-3r^2+8Rr\cos A) \therefore (\blacksquare),(\blacksquare\blacksquare)$$

$$\Rightarrow p_a = \frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}$$

$$\Rightarrow \frac{1}{p_a} = \frac{1}{2s}.\frac{2s+a}{\sqrt{s^2-8Rr-3r^2+8Rr\cos A}} \text{ and analogs} \Rightarrow \frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}$$

$$= \frac{1}{2s}.\sum_{\text{cyc}} \frac{(2s+a)^2}{\sqrt{s^2-8Rr-3r^2+8Rr\cos A}.(2s+a)}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{1}{2s}.\frac{(6s+2s)^2}{\sum_{\text{cyc}}\left(\sqrt{s^2-8Rr-3r^2+8Rr\cos A}.(2s+a)\right)}$$

$$= \frac{1}{2s}.\frac{64s^2}{\sum_{\text{cyc}}\left(\sqrt{(s^2-8Rr-3r^2+8Rr\cos A)(2s+a)}.\sqrt{2s+a}\right)}$$

$$\overset{\text{CBS}}{\geq} \frac{1}{2s}.\frac{64s^2}{\sqrt{\sum_{\text{cyc}}((s^2-8Rr-3r^2+8Rr\cos A)(2s+a))}.\sqrt{6s+2s}}$$

$$= \frac{1}{2s}.\frac{64s^2}{\sqrt{(s^2-8Rr-3r^2)(8s)+16Rrs\left(\frac{R+r}{R}\right)+8Rr.\frac{2rs}{R}}.\sqrt{8s}}$$

$$\left(\because \sum_{\text{cyc}} a\cos A = \frac{2rs}{R}\right) = \frac{1}{2s}.\frac{64s^2}{8s.\sqrt{s^2-6Rr+r^2}} \overset{\text{Gerretsen}}{\geq}$$

$$\frac{4}{\sqrt{4R^2+4Rr+3r^2-6Rr+r^2}} = \frac{4}{\sqrt{4R^2-2Rr+4r^2}} \overset{\text{Euler}}{\geq} \frac{4}{\sqrt{4R^2-2Rr+2Rr}} = \frac{2}{R}$$

$$\therefore \frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c} \geq \frac{2}{R}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1583. In ΔABC the following relationships holds :

$$\sqrt[5]{\frac{s_a}{m_a}}+\sqrt[5]{\frac{s_b}{m_b}}+\sqrt[5]{\frac{s_c}{m_c}}+\frac{r_ar_br_c}{w_aw_bw_c} \geq 4$$

Proposed by Adil Abdullayev-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have } \frac{w_a}{\sqrt{r_b r_c}} = \frac{2\sqrt{bc}}{b+c} \le 1 \text{ (and analogs).}$$

By using the AM − GM Inequality, we have

$$\frac{s_a}{m_a} = \frac{2bc}{b^2+c^2} = \frac{16b^2c^2}{4.2bc.(b^2+c^2)} \ge \frac{16b^2c^2}{[2bc+(b^2+c^2)]^2} = \left(\frac{2\sqrt{bc}}{b+c}\right)^4 \ge$$

$$\ge \left(\frac{2\sqrt{bc}}{b+c}\right)^5 = \left(\frac{w_a}{\sqrt{r_b r_c}}\right)^5 \Rightarrow \sqrt[5]{\frac{s_a}{m_a}} \ge \frac{w_a}{\sqrt{r_b r_c}} \text{ (and analogs)}$$

Therefore

$$\sqrt[5]{\frac{s_a}{m_a}} + \sqrt[5]{\frac{s_b}{m_b}} + \sqrt[5]{\frac{s_c}{m_c}} + \frac{r_a r_b r_c}{w_a w_b w_c} \ge \frac{w_a}{\sqrt{r_b r_c}} + \frac{w_b}{\sqrt{r_c r_a}} + \frac{w_c}{\sqrt{r_a r_b}} + \frac{r_a r_b r_c}{w_a w_b w_c}$$

$$\overset{AM-GM}{\ge} 4\sqrt[4]{\frac{w_a}{\sqrt{r_b r_c}} \cdot \frac{w_b}{\sqrt{r_c r_a}} \cdot \frac{w_c}{\sqrt{r_a r_b}} \cdot \frac{r_a r_b r_c}{w_a w_b w_c}} = 4.$$

Equality holds iff ΔABC is equilateral.

1584.

In any Δ ABC, the following relationship holds :

$$\frac{m_a m_b m_c}{r_a r_b r_c} \le \frac{R}{2r}$$

Proposed by Adil Abdullayev-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64}(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)$$

$$\overset{(1)}{=} \frac{1}{64}\left\{-4\sum_{cyc} a^6 + 6\left(\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4\right) + 3a^2b^2c^2\right\}$$

$$\text{Now, } \sum_{cyc} a^6 = \left(\sum_{cyc} a^2\right)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= \left(\sum_{cyc} a^2\right)^3 - 3\left(2a^2b^2c^2 + \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right)\right)$$

$$= \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\therefore \sum_{cyc} a^6 \overset{(2)}{=} \left(\sum_{cyc} a^2\right)^3 + 3a^2b^2c^2 - 3\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right)$$

$$\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 = \sum_{cyc}\left(a^2b^2\left(\sum_{cyc} a^2 - c^2\right)\right) \overset{(3)}{=}$$

$$\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 3a^2b^2c^2 \therefore (1),(2),(3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64}\left(\begin{matrix} -4\left(\sum_{cyc} a^2\right)^3 - 12a^2b^2c^2 + 12\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) \\ +6\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 18a^2b^2c^2 + 3a^2b^2c^2 \end{matrix}\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\sum_{cyc} a^2 b^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left(-4\left(\sum_{cyc} a^2\right)^3 + 18\left(\left(\sum_{cyc} ab\right)^2 - 16Rrs^2\right)\left(\sum_{cyc} a^2\right) - 27a^2b^2c^2\right)$$

$$= \frac{1}{64}\left\{\begin{matrix} -32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2)^2 \\ -576Rrs^2(s^2 - 4Rr - r^2) - 432R^2r^2s^2 \end{matrix}\right\}$$

$$= \frac{1}{16}\{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3\}$$

$$\le \frac{R^2s^4}{4} \Leftrightarrow$$

$$s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \overset{(\bullet)}{\le} 0$$

$$\text{Now, LHS of } (\bullet) \overset{\text{Gerretsen}}{\le} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4)$$

$$-r^3(4R + r)^3 \overset{?}{\le} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \underset{(\bullet\bullet)}{\overset{?}{\ge}} 20rs^4$$

$$\text{Now, LHS of } (\bullet\bullet) \underbrace{\overset{\text{Gerretsen}}{\ge}}_{(a)} s^2(16Rr - 5r^2)(8R - 16r)$$

$$+s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \text{ and}$$

$$\text{RHS of } (\bullet\bullet) \underbrace{\overset{\text{Geretsen}}{\le}}_{(b)} 20rs^2(4R^2 + 4Rr + 3r^2)$$

$(a),(b) \Rightarrow$ **in order to prove** $(\bullet\bullet)$, **it suffices to prove :**

$$s^2(16Rr-5r^2)(8R-16r)+s^2(60R^2r+120Rr^2+33r^3)+r^2(4R+r)^3$$
$$\ge 20rs^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow s^2(108R^2-256Rr+53r^2)+r(4R+r)^3\ge 0$$
$$\Leftrightarrow s^2(108R^2-256Rr+80r^2)+r(4R+r)^3 \overset{(\bullet\bullet\bullet)}{\ge} 27r^2s^2$$

Now, LHS of $(\bullet\bullet\bullet)\underbrace{\overset{Gerretsen}{\ge}}_{(c)}(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3$

and RHS of $(\bullet\bullet\bullet)\underbrace{\overset{Geretsen}{\le}}_{(d)}27r^2(4R^2+4Rr+3r^2)$

$(c),(d)\Rightarrow$ in order to prove $(\bullet\bullet\bullet)$, it suffices to prove :

$$(108R^2-256Rr+80r^2)(16Rr-5r^2)+r(4R+r)^3\ \ge 27r^2(4R^2+4Rr+3r^2)$$
$$\Leftrightarrow 224t^3-587t^2+308t-60\ge 0\ \left(\text{where } t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\big((t-2)(224t+309)+648\big)\ge 0\to \text{true} \because t\overset{Euler}{\ge}2\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)$$
$$\Rightarrow(\bullet)\text{ is true}\Rightarrow m_a^2m_b^2m_c^2\le\frac{R^2s^4}{4}\Rightarrow m_am_bm_c\le\frac{Rs^2}{2}\Rightarrow\frac{m_am_bm_c}{r_ar_br_c}\le\frac{\frac{Rs^2}{2}}{rs^2}$$
$$\Rightarrow\frac{m_am_bm_c}{r_ar_br_c}\le\frac{R}{2r}\ \forall\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1585. In any ΔABC, the following relationships hold :

$$\sqrt{r_br_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le\csc\frac{A}{2}\ \text{and}\ \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{r_br_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\overset{Lascu+A-G}{\le}\sqrt{s(s-a)}\left(\frac{1}{\sqrt{s(s-b)}}+\frac{1}{\sqrt{s(s-c)}}\right)\overset{?}{\le}\csc\frac{A}{2}$$
$$\Leftrightarrow\sqrt{\frac{s-a}{s-b}}+\sqrt{\frac{s-a}{s-c}}\overset{?}{\le}\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}}$$
$$\Leftrightarrow\sqrt{(s-a)(s-c)}+\sqrt{(s-a)(s-b)}\overset{?}{\le}\sqrt{bc}\Leftrightarrow\sqrt{xz}+\sqrt{xy}\underbrace{\overset{?}{\le}}_{(*)}\sqrt{(z+x)(x+y)}$$
$$\left(\text{where } s-a=x, s-b=y, s-c=z\Rightarrow s=x+y+z\Rightarrow a=y+z, b=z+x,\ c=x+y\right)$$

Now, $(z+x)(x+y)\overset{Reverse\ CBS}{\ge}\left(\sqrt{xz}+\sqrt{xy}\right)^2\Rightarrow(*)$ is true

$$\therefore\boxed{\sqrt{r_br_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le\csc\frac{A}{2}}\Rightarrow\frac{1}{m_b}+\frac{1}{m_c}\le\frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}.\sqrt{s(s-a)}}$$
$$\Rightarrow\frac{1}{m_b}+\frac{1}{m_c}\le\frac{\sqrt{bc}}{F}\to(1)$$

$$\text{Now,}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\Leftrightarrow\left(\frac{b+c}{\sqrt{bc}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2.\frac{b+c}{2F}$$

$$\Leftrightarrow\frac{1}{m_b}+\frac{1}{m_c}\le\frac{\sqrt{bc}}{F}\to\text{true via }(1)\ \therefore\ \boxed{\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right)}$$

$$\therefore\sqrt{r_br_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le\csc\frac{A}{2}\ \text{and}\ \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le 2\left(\frac{1}{h_b}+\frac{1}{h_c}\right),$$

$''=''$ iff Δ ABC is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Tereshin and CBS inequalities, we have

$$\sqrt{m_bm_c}\ge\sqrt{\frac{(c^2+a^2)(a^2+b^2)}{4R.4R}}\ge\frac{ac+ab}{4R}=\frac{abc}{4R}\left(\frac{1}{b}+\frac{1}{c}\right)=F\left(\frac{1}{b}+\frac{1}{c}\right).$$

We know that m_a, m_b, m_c can be the sides of a triangle with area $F'=\frac{3F}{4}$ and medians $m'_a=\frac{3a}{4}$, $m'_b=\frac{3b}{4}, m'_c=\frac{3c}{4}$. By using the last inequality in this triangle, we have

$$\sqrt{m'_bm'_c}\ge F'\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\ \Rightarrow\ \frac{1}{m_b}+\frac{1}{m_c}\le\frac{\sqrt{bc}}{F}.$$

Using this result, we have

$$\sqrt{r_br_c}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le\sqrt{\frac{F^2}{(s-b)(s-b)}}.\frac{\sqrt{bc}}{F}=\csc\frac{A}{2},$$

$$\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\le\frac{b+c}{\sqrt{bc}}.\frac{\sqrt{bc}}{F}=2\left(\frac{1}{h_b}+\frac{1}{h_c}\right),$$

as desired. Equality holds iff ΔABC is equilateral.

1586. **Prove that:**

$$(x+y+z)(xAP^2+yBP^2+zCP^2) \geq yza^2+zxb^2+xyc^2+(x+y+z)^2{h_p}^2$$

for P a point in space,
h_p perpendicular from P on (ABC), x,y,z real numbers, a,b,c sides of ΔABC

Proposed by Bogdan Fuștei-Romania
Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let M be the feet of the perpendicular from P to (ABC).

$$\text{We have } \left(x\overrightarrow{AM}+y\overrightarrow{BM}+z\overrightarrow{CM}\right)^2 \geq 0$$

$$\Leftrightarrow\ x^2AM^2+y^2BM^2+z^2CM^2+2xy\overrightarrow{AM}.\overrightarrow{BM}+2yz\overrightarrow{BM}.\overrightarrow{CM}+2zx\overrightarrow{CM}.\overrightarrow{AM} \geq 0.$$

$$\text{Since } 2\overrightarrow{BM}.\overrightarrow{CM} = BM^2+CM^2-BC^2 \text{ (and analogs), and}$$

$$AM^2 = AP^2-{h_p}^2 \text{ (and analogs), then}$$

$$x^2\left(AP^2-{h_p}^2\right)+y^2\left(BP^2-{h_p}^2\right)+z^2\left(CP^2-{h_p}^2\right)+xy\left(AP^2+BP^2-2{h_p}^2-c^2\right)+$$

$$+yz\left(BP^2+CP^2-2{h_p}^2-a^2\right)+zx\left(CP^2+AP^2-2{h_p}^2-b^2\right) \geq 0$$

$$\Leftrightarrow (x+y+z)(xAP^2+yBP^2+zCP^2) \geq yza^2+zxb^2+xyc^2+(x+y+z)^2{h_p}^2,$$

as desired. Equality holds if and only if the barycentric coordinates of point M is (x,y,z).

1587. **In ΔABC holds :**

$$\frac{m_a}{\sqrt{m_bm_c}}+\frac{m_b}{\sqrt{m_cm_a}}+\frac{m_c}{\sqrt{m_am_b}} \leq 3\sqrt{\frac{R}{2r}}$$

Proposed by George Apostolopoulos-Messolonghi-Greece
Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let S be the area of ΔABC. By CBS inequality, we have

$$\frac{m_a}{\sqrt{m_bm_c}}+\frac{m_b}{\sqrt{m_cm_a}}+\frac{m_c}{\sqrt{m_am_b}} \leq \sqrt{({m_a}^2+{m_b}^2+{m_c}^2)\left(\frac{1}{m_bm_c}+\frac{1}{m_cm_a}+\frac{1}{m_am_b}\right)}.$$

So it suffices to prove that

$$(m_a{}^2+m_b{}^2+m_c{}^2)\left(\frac{1}{m_bm_c}+\frac{1}{m_cm_a}+\frac{1}{m_am_b}\right)\le\frac{9R}{2r}=\frac{9abc(a+b+c)}{16S^2}.$$

We know that m_a, m_b, m_c can be the sides of a triangle with area S' $=\frac{3S}{4}$ and medians $m'_a=\frac{3a}{4}$, $m'_b=\frac{3b}{4}, m'_c=\frac{3c}{4}$. Then the inequality we have to prove is

$$(m_a{}^2+m_b{}^2+m_c{}^2)\left(\frac{1}{m_bm_c}+\frac{1}{m_cm_a}+\frac{1}{m_am_b}\right)\le\frac{m'_am'_bm'_c(m'_a+m'_b+m'_c)}{S'^2}.$$

Now, the last inequality will be true if the triangle with side − lengths a, b, c and area S satisfies the following inequality :

$$(a^2+b^2+c^2)\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)\le\frac{m_am_bm_c(m_a+m_b+m_c)}{S^2}.$$

By Tereshin's inequality, we have

$$m_a+m_b+m_c\ge\frac{b^2+c^2}{4R}+\frac{c^2+a^2}{4R}+\frac{a^2+b^2}{4R}=\frac{a^2+b^2+c^2}{2R}.$$

Using the known inequality, $m_a\ge\sqrt{s(s-a)}$ (and analogs), we have

$$m_am_bm_c\ge\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}=\frac{S^2}{r}.$$

Then

$$\frac{m_am_bm_c(m_a+m_b+m_c)}{S^2}\ge\frac{a^2+b^2+c^2}{2Rr}=(a^2+b^2+c^2)\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right),$$

which completes the proof. Equality holds iff ΔABC is equilateral.

1588. In ΔABC holds:

$$\sum_{cyc}\sin^4 A.\sin(2A)\le\frac{27\sqrt{3}}{32}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $sin\,A=\frac{a}{2R}$ and $\cos A=\frac{b^2+c^2-a^2}{2bc}$ (and analogs), where R is the circumradius of ΔABC, then we have

$$\sum_{cyc}\sin^4 A.\sin(2A)=\sum_{cyc}\sin^5 A.2\cos A=\sum_{cyc}\frac{a^5(b^2+c^2-a^2)}{32R^5.bc}=$$

$$= \frac{1}{32R^5}\sum_{cyc}\frac{a^3[b^2c^2-(a^2-b^2)(a^2-c^2)]}{bc}$$

$$= \frac{abc(a^2+b^2+c^2)}{32R^5} - \frac{\sum_{cyc}a^4(a^2-b^2)(a^2-c^2)}{32R^5.abc}.$$

By Schur's inequality, we have

$$\sum_{cyc}a^4(a^2-b^2)(a^2-c^2) \geq 0.$$

By Leibniz's inequality, we have

$a^2+b^2+c^2 \leq 9R^2$, and by Mitrinovic and Euler inequalities, we have $abc = R.2s.2r \leq R.3\sqrt{3}R.R = 3\sqrt{3}R^3$.

Using these results, we have

$$\sum_{cyc}\sin^4 A.\sin(2A) \leq \frac{abc(a^2+b^2+c^2)}{32R^5} \leq \frac{3\sqrt{3}R^3.9R^2}{32R^5} = \frac{27\sqrt{3}}{32},$$

as desired. Equality holds iff ΔABC is equilateral.

1589. **In any Δ ABC, the following relationship holds :**

$$3 \leq \sum_{cyc}\frac{m_a h_a}{h_b h_c} \leq 3\left(\frac{R}{2r}\right)^4$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a h_a}{h_b h_c} = \sum_{cyc}\frac{m_a \frac{bc}{2R}}{\frac{ca.ab}{4R^2}} = 2R\sum_{cyc}\frac{m_a^2}{m_a a^2} \overset{\text{Tereshin}}{\leq} 8R^2\sum_{cyc}\frac{m_a^2}{a^2(b^2+c^2)}$$

$$= 2R^2\sum_{cyc}\frac{2b^2+2c^2-a^2}{a^2(b^2+c^2)} = 4R^2.\frac{\sum_{cyc}b^2c^2}{a^2b^2c^2} - 2R^2\sum_{cyc}\frac{1}{b^2+c^2} \overset{\text{Goldstone and Bergstrom}}{\leq}$$

$$\frac{4R^2.4R^2s^2}{16R^2r^2s^2} - 2R^2.\frac{9}{2\sum_{cyc}a^2} \overset{\text{Leibnitz}}{\leq} \frac{R^2}{r^2} - \frac{2R^2.9}{2.9R^2} = \frac{R^2}{r^2} - 1 \overset{?}{\leq} 3\left(\frac{R}{2r}\right)^4$$

$$\Leftrightarrow 3t^4 \overset{?}{\geq} 16t^2 - 16 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left((t-2)(3t^2+12t+20)+32\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \therefore \sum_{cyc}\frac{m_a h_a}{h_b h_c} \leq 3\left(\frac{R}{2r}\right)^4$$

$$\text{Again, }\sum_{cyc}\frac{m_a h_a}{h_b h_c} = 2R\sum_{cyc}\frac{m_a}{a^2} \overset{\text{Tereshin}}{\geq} \frac{2R}{4R}\sum_{cyc}\frac{b^2+c^2}{a^2} = \frac{1}{2}\sum_{cyc}\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right) \overset{A-G}{\geq} \frac{1}{2}.6$$

$$\therefore \sum_{cyc}\frac{m_a h_a}{h_b h_c}\geq 3 \therefore 3\leq \sum_{cyc}\frac{m_a h_a}{h_b h_c}\leq 3\left(\frac{R}{2r}\right)^4$$

$\forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC$ is equilateral (QED)

1590. In any $\Delta\,ABC$, the following relationship holds :

$$\sum_{cyc}\frac{2h_b h_c}{9r^2}\geq 3+9r^2\sum_{cyc}\frac{1}{h_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{2h_b h_c}{9r^2}\geq 3+9r^2\sum_{cyc}\frac{1}{h_a^2}\Leftrightarrow \sum_{cyc}\frac{2ca.ab}{4R^2.9r^2}\geq 3+\frac{9r^2}{4r^2s^2}\sum_{cyc}a^2$$

$$\Leftrightarrow \frac{2.4Rrs.2s}{4R^2.9r^2}\geq 3+\frac{9(s^2-4Rr-r^2)}{2s^2}\Leftrightarrow \frac{4s^2-27Rr}{9Rr}\geq \frac{9(s^2-4Rr-r^2)}{2s^2}$$

$$\Leftrightarrow 8s^4-135Rrs^2+81Rr^2(4R+r)\overset{(*)}{\geq} 0 \text{ and } \because 8(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\geq 8(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (121R-80r)s^2\overset{(**)}{\geq} r(1724R^2-1361Rr+200r^2)$$

$$\text{Now, LHS of } (**)\overset{\text{Gerretsen}}{\geq}(121R-80r)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(1724R^2-1361Rr+200r^2)\Leftrightarrow 53R^2-131Rr+50r^2\overset{?}{\geq}0$$

$$\Leftrightarrow (R-2r)(53R-25r)\overset{?}{\geq}0 \to \text{true} \because R\overset{\text{Euler}}{\geq}2r\Rightarrow (**)\Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{2h_b h_c}{9r^2}\geq 3+9r^2\sum_{cyc}\frac{1}{h_a^2}\ \forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1591. In any $\Delta\,ABC$, the following relationship holds :

$$F\left(4-\frac{2r}{R}\right)^2\leq \sum_{cyc}h_a^2\cot\frac{A}{2}\leq 9F\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}h_a^2\cot\frac{A}{2}=\sum_{cyc}\left(\frac{b^2c^2}{4R^2}.\frac{s-a}{r}\right)$$

$$=\frac{s\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-4Rrs(s^2+4Rr+r^2)}{4R^2r}$$

$$\Rightarrow \sum_{cyc} h_a^2 \cot\frac{A}{2} = \frac{s\left(s^4-(12Rr-2r^2)s^2+r^3(4R+r)\right)}{4R^2r} \to (1)$$

$$\therefore \text{ via } (1), \sum_{cyc} h_a^2 \cot\frac{A}{2} \le 9F\left(\frac{R}{2r}\right)^3$$

$$\Leftrightarrow \frac{s\left(s^4-(12Rr-2r^2)s^2+r^3(4R+r)\right)}{4R^2r} \le \frac{9rsR^3}{8r^3}$$

$$\Leftrightarrow 9R^5-8R^4r-2r^5+4r^2s^2(6R-r)-2rs^4 \underset{}{\overset{(*)}{\ge}} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\ge} 9R^5-8R^4r-2r^5+4r^2s^2(6R-r)$$

$$-2rs^2(4R^2+4Rr+3r^2) \overset{?}{\ge} 0 \Leftrightarrow 2rs^2(4R^2-8Rr+5r^2) \underset{(**)}{\overset{?}{\le}} 9R^5-8R^4r-2r^5$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\le} 2r(4R^2-8Rr+5r^2)(4R^2+4Rr+3r^2) \overset{?}{\le}$$

$$9R^5-8R^4r-2r^5 \Leftrightarrow 9t^5-32t^4+32t^3-32 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(7t^3(t-2)+2t^4+4t^2+8t+16\right) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} h_a^2 \cot\frac{A}{2} \le 9F\left(\frac{R}{2r}\right)^3$$

$$\text{Also, via } (1), \sum_{cyc} h_a^2 \cot\frac{A}{2} \ge F\left(4-\frac{2r}{R}\right)^2$$

$$\Leftrightarrow \frac{s\left(s^4-(12Rr-2r^2)s^2+r^3(4R+r)\right)}{4R^2r} \ge \frac{rs(4R-2r)^2}{R^2}$$

$$\Leftrightarrow s^4-(12Rr-2r^2)s^2-r^2(64R^2-68Rr+15r^2) \underset{(***)}{\overset{?}{\ge}} 0$$

$$\text{Now, } s^4-(12Rr-2r^2)s^2 \overset{\text{Gerretsen}}{\ge} (16Rr-5r^2)s^2-(12Rr-2r^2)s^2$$

$$=(4Rr-3r^2)s^2 \overset{\text{Gerretsen}}{\ge} (4Rr-3r^2)(16Rr-5r^2)=r^2(64R^2-68Rr+15r^2)$$

$$\Rightarrow (***) \text{ is true } \therefore \sum_{cyc} h_a^2 \cot\frac{A}{2} \ge F\left(4-\frac{2r}{R}\right)^2 \text{ and so,}$$

$$F\left(4-\frac{2r}{R}\right)^2 \le \sum_{cyc} h_a^2 \cot\frac{A}{2} \le 9F\left(\frac{R}{2r}\right)^3 \ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1592. In ΔABC the following relationship holds:

$$\frac{a^2}{(b+c)^2-a^2}+\frac{b^2}{(c+a)^2-b^2}+\frac{c^2}{(a+b)^2-c^2} \ge 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{a^2}{(b+c)^2-a^2}=\sum_{cyc}\frac{a^2}{(b+c+a)(b+c-a)}=$$

$$=\sum_{cyc}\frac{a^2}{2s(2s-2a)}=\frac{1}{4}\sum_{cyc}\frac{a^2}{s(s-a)}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{1}{4}\cdot\frac{(a+b+c)^2}{s(s-a)+s(s-b)+s(s-c)}=$$

$$=\frac{1}{4}\cdot\frac{4s^2}{s(s-a+s-b+s-c)}=\frac{s}{3s-a-b-c}=\frac{s}{3s-2s}=1$$

Equality holds for $a=b=c$.

1593.

In any Δ ABC, the following relationship holds :

$$\frac{m_a^5+m_b^5}{m_a^4m_b^4(m_a+m_b)}+\frac{m_b^5+m_c^5}{m_b^4m_c^4(m_b+m_c)}+\frac{m_c^5+m_a^5}{m_c^4m_a^4(m_c+m_a)}\geq\frac{16}{27R^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, x,y>0,\frac{x^5+y^5}{x^4y^4(x+y)}=\frac{(x+y)\left((x^4+x^2y^2+y^4)-(x^3y+xy^3)\right)}{x^4y^4(x+y)}=$$

$$\frac{(x^2+y^2)^2-x^2y^2-xy(x^2+y^2)}{x^4y^4}=\frac{(x^2+y^2+xy)(x^2+y^2-xy)-xy(x^2+y^2)}{x^4y^4}$$

$$\overset{A-G}{\geq}\frac{xy(x^2+y^2+xy)-xy(x^2+y^2)}{x^4y^4}=\frac{1}{x^2y^2}\Rightarrow\frac{x^5+y^5}{x^4y^4(x+y)}\geq\frac{1}{x^2y^2}\to(1)$$

$$\text{and analogously},\frac{y^5+z^5}{y^4z^4(y+z)}\geq\frac{1}{y^2z^2}\to(2)\text{ and }\frac{z^5+x^5}{z^4x^4(z+x)}\geq\frac{1}{z^2x^2}\to(3)$$

and via $x\equiv m_a, y\equiv m_b, z\equiv m_c$, we arrive at :

$$\frac{m_a^5+m_b^5}{m_a^4m_b^4(m_a+m_b)}+\frac{m_b^5+m_c^5}{m_b^4m_c^4(m_b+m_c)}+\frac{m_c^5+m_a^5}{m_c^4m_a^4(m_c+m_a)}\geq\sum_{cyc}\frac{1}{m_a^2m_b^2}\overset{Bergstrom}{\geq}$$

$$\frac{9}{\sum_{cyc}m_a^2m_b^2}=\frac{9}{\frac{9}{16}\sum_{cyc}a^2b^2}\overset{Goldstone}{\geq}\frac{16}{4R^2s^2}\overset{Mitrinovic}{\geq}\frac{16}{4R^2s^2}=\frac{16}{R^2.27R^2}$$

$$\therefore \frac{m_a^5+m_b^5}{m_a^4m_b^4(m_a+m_b)}+\frac{m_b^5+m_c^5}{m_b^4m_c^4(m_b+m_c)}+\frac{m_c^5+m_a^5}{m_c^4m_a^4(m_c+m_a)}\ge\frac{16}{27R^4}$$

$\forall\ \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1594.

In any ΔABC, the following relationship holds:

$$\frac{r_a^5+r_b^5}{r_a^3r_b^3(r_a^2+r_b^2)}+\frac{r_b^5+r_c^5}{r_b^3r_c^3(r_b^2+r_c^2)}+\frac{r_c^5+r_a^5}{r_c^3r_a^3(r_c^2+r_a^2)}\ge\frac{16r}{9R^4}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x^5+y^5}{x^3y^3(x^2+y^2)}=\frac{(x^2+y^2)(x^3+y^3)-x^2y^2(x+y)}{x^3y^3(x^2+y^2)}$$

$$=\frac{x^3+y^3}{x^3y^3}-\frac{x+y}{xy(x^2+y^2)}\ge\frac{xy(x+y)}{x^3y^3}-\frac{2(x+y)}{xy(x+y)^2}=\frac{x+y}{x^2y^2}-\frac{2}{xy(x+y)}$$

$$=\frac{(x+y)^2-2xy}{x^2y^2(x+y)}=\frac{x^2+y^2}{x^2y^2(x+y)}\ge\frac{(x+y)^2}{2x^2y^2(x+y)}\Rightarrow\frac{x^5+y^5}{x^3y^3(x^2+y^2)}\ge\frac{x+y}{2x^2y^2}$$

$\to(1)$ and analogously, $\dfrac{y^5+z^5}{y^3z^3(y^2+z^2)}\ge\dfrac{y+z}{2y^2z^2}\to(2)$ and $\dfrac{z^5+x^5}{z^3x^3(z^2+x^2)}\ge\dfrac{z+x}{2z^2x^2}$

$\to(3)$ and via $x\equiv r_a, y\equiv r_b, z\equiv r_c$, we arrive at:

$$\frac{r_a^5+r_b^5}{r_a^3r_b^3(r_a^2+r_b^2)}+\frac{r_b^5+r_c^5}{r_b^3r_c^3(r_b^2+r_c^2)}+\frac{r_c^5+r_a^5}{r_c^3r_a^3(r_c^2+r_a^2)}\ge\sum_{cyc}\frac{r_a+r_b}{2r_a^2r_b^2}$$

$$=\frac{1}{2r_a^2r_b^2r_c^2}\cdot\sum_{cyc}r_c^2(r_a+r_b)=\frac{1}{2r^2s^4}\cdot\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-3r_ar_br_c\right)$$

$$=\frac{(4R+r)s^2-3rs^2}{2r^2s^4}=\frac{2R-r}{r^2s^2}\overset{\substack{\text{Euler}\\ \text{and}\\ \text{Mitrinovic}}}{\ge}\frac{3r}{\frac{R^2}{4}\cdot\frac{27R^2}{4}}$$

$$\therefore\frac{r_a^5+r_b^5}{r_a^3r_b^3(r_a^2+r_b^2)}+\frac{r_b^5+r_c^5}{r_b^3r_c^3(r_b^2+r_c^2)}+\frac{r_c^5+r_a^5}{r_c^3r_a^3(r_c^2+r_a^2)}\ge\frac{16r}{9R^4}\ \forall\ \Delta ABC,$$

" = " iff ΔABC is equilateral (QED)

1595.

In ΔABC the following relationships holds:

$$\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}+\frac{R^2}{4r^2}\ge 1+\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $m_a \geq \frac{b+c}{2}\cos\frac{A}{2} = \frac{2(s-a)+a}{2}\cdot\sqrt{\frac{s(s-a)}{bc}} \geq$

$$\geq \sqrt{2(s-a)a}.\sqrt{\frac{a(s-a)}{4Rr}} = \frac{a(s-a)}{\sqrt{2Rr}} \Rightarrow \frac{m_a}{r_a} \geq \frac{a(s-a)^2}{F\sqrt{2Rr}} \text{ (and analogs)}$$

$$\Rightarrow \frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c} \geq \frac{a(s-a)^2+b(s-b)^2+c(s-c)^2}{F\sqrt{2Rr}}$$

$$= \frac{s^2.2s-2s.2(s^2-r^2-4Rr)+2s(s^2-3r^2-6Rr)}{F\sqrt{2Rr}} = \frac{4R-2r}{\sqrt{2Rr}}.$$

Now, since $m_a \geq h_a$ (and analogs), then we have

$$\frac{r_a}{m_a}+\frac{r_b}{m_b}+\frac{r_c}{m_c} \leq \frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c} = \sum_{cyc}\frac{a}{2(s-a)} = \frac{1}{2}\sum_{cyc}\left(\frac{s}{s-a}-1\right) =$$

$$= \frac{1}{2}\left(\frac{4R+r}{r}-3\right) = \frac{2R}{r}-1.$$

From these results, it suffices to prove that

$$\frac{4R-2r}{\sqrt{2Rr}}+\frac{R^2}{4r^2} \geq \frac{2R}{r} \overset{x=\sqrt{\frac{R}{2r}}}{\Leftrightarrow} 4x-\frac{1}{x}+x^4 \geq 4x^2 \Leftrightarrow x^5-4x^3+4x^2-1 \geq 0$$

$$\Leftrightarrow (x-1)[x(x-1)(x^2+2x-1)+1] \geq 0,$$

which is true by Euler's inequality

$x = \sqrt{\frac{R}{2r}} \geq 1$. Equality holds iff ΔABC is equilateral.

1596. **If $a,b,c > 0, n \in \mathbb{N}$ then**

$$\frac{{h_a}^n({h_a}^2+w_bm_c)}{{w_b}^{n+1}+{m_c}^{n+1}}+\frac{{w_b}^n({w_b}^2+m_ch_a)}{{m_c}^{n+1}+{h_a}^{n+1}}+\frac{{m_c}^n({m_c}^2+h_aw_b)}{{h_a}^{n+1}+{w_b}^{n+1}} \geq 9r$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will first prove that for all $x,y,z > 0$, the following inequality holds:

$$\frac{x^n(x^2+yz)}{y^{n+1}+z^{n+1}}+\frac{y^n(y^2+zx)}{z^{n+1}+x^{n+1}}+\frac{z^n(z^2+xy)}{x^{n+1}+y^{n+1}} \geq x+y+z.$$

The last inequality can be rewritten as follows

$$\sum_{cyc}\left(\frac{x^n(x-y)(x-z)}{y^{n+1}+z^{n+1}}+\frac{x^{n+1}(y+z)}{y^{n+1}+z^{n+1}}\right) \geq x+y+z.$$

Notice that if $x \geq y \geq z$ then

$$\frac{x^n}{y^{n+1}+z^{n+1}} \geq \frac{y^n}{z^{n+1}+x^{n+1}} \geq \frac{z^n}{x^{n+1}+y^{n+1}},$$

so by the Generalized Schur inequality, we deduce that

$$\sum_{cyc} \frac{x^n}{y^{n+1}+z^{n+1}}(x-y)(x-z) \geq 0.$$

So it suffices to prove that

$$\sum_{cyc} \frac{x^{n+1}(y+z)}{y^{n+1}+z^{n+1}} \geq x+y+z \Leftrightarrow \sum_{cyc}\left(\frac{x^{n+1}(y+z)}{y^{n+1}+z^{n+1}}-x\right) \geq 0$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{xy(x^n-y^n)}{y^{n+1}+z^{n+1}}-\frac{zx(z^n-x^n)}{y^{n+1}+z^{n+1}}\right) \geq 0 \Leftrightarrow \sum_{cyc}\left(\frac{xy(x^n-y^n)}{y^{n+1}+z^{n+1}}-\frac{xy(x^n-y^n)}{z^{n+1}+x^{n+1}}\right) \geq 0$$

$$\Leftrightarrow \sum_{cyc} \frac{xy(x^n-y^n)(x^{n+1}-y^{n+1})}{(y^{n+1}+z^{n+1})(z^{n+1}+x^{n+1})} \geq 0,$$

which is true because x^n-y^n and $x^{n+1}-y^{n+1}$ have the same sign.
Taking now $x=h_a, \ y=w_b, \ z=m_c$. We have

$$\frac{{h_a}^n({h_a}^2+w_b m_c)}{{w_b}^{n+1}+{m_c}^{n+1}}+\frac{{w_b}^n({w_b}^2+m_c h_a)}{{m_c}^{n+1}+{h_a}^{n+1}}+\frac{{m_c}^n({m_c}^2+h_a w_b)}{{h_a}^{n+1}+{w_b}^{n+1}} \geq h_a+w_b+m_c$$

$$\overset{w_b \geq h_b \,\&\, m_c \geq h_a}{\geq} h_a+h_b+h_c \overset{AM-HM}{\geq} \frac{9}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}} = 9r,$$

as desired. Equality holds iff ΔABC is equilateral.

1597. In any Δ ABC, the following relationship holds :

$$\sum_{cyc} h_a^2 \cot\frac{A}{2} \geq \frac{2r}{R}\sum_{cyc} r_a^2 \cot\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} h_a^2 \cot\frac{A}{2} = \sum_{cyc}\left(\frac{b^2c^2}{4R^2}\cdot\frac{s-a}{r}\right)$$

$$= \frac{s\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-4Rrs\left(s^2+4Rr+r^2\right)}{4R^2r}$$

$$\Rightarrow \sum_{cyc} h_a^2 \cot\frac{A}{2} = \frac{s\left(s^4-\left(12Rr-2r^2\right)s^2+r^3(4R+r)\right)}{4R^2r} \rightarrow (1)$$

$$\text{Also, }\sum_{cyc} r_a^2 \cot\frac{A}{2} = \sum_{cyc}\left(\frac{r^2s^2}{(s-a)^2}\cdot\frac{s-a}{r}\right) = s\sum_{cyc} r_a$$

$$\Rightarrow \sum_{cyc} r_a^2 \cot\frac{A}{2} = s(4R+r) \to (2) \therefore (1),(2) \Rightarrow \sum_{cyc} h_a^2 \cot\frac{A}{2} \ge \frac{2r}{R}\sum_{cyc} r_a^2\cot\frac{A}{2}$$

$$\Leftrightarrow \frac{s\left(s^4-(12Rr-2r^2)s^2+r^3(4R+r)\right)}{4R^2r} \ge \frac{2r}{R}.s(4R+r)$$

$$\Leftrightarrow s^4-(12Rr-2r^2)s^2-r^2(32R^2+4Rr-15r^2) \overset{(*)}{\ge} 0$$

$$\text{Now, LHS of }(*) \overset{\text{Gerretsen}}{\ge} (16Rr-5r^2)s^2-(12Rr-2r^2)s^2$$

$$-r^2(32R^2+4Rr-15r^2) = (4Rr-3r^2)s^2 - r^2(32R^2+4Rr-15r^2)$$

$$\overset{\text{Gerretsen}}{\ge} (4Rr-3r^2)(16Rr-5r^2)-r^2(32R^2+4Rr-15r^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow 4R^2-9Rr+2r^2 \overset{?}{\ge} 0 \Leftrightarrow (R-2r)(4R-r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (*) \text{ is true}$$

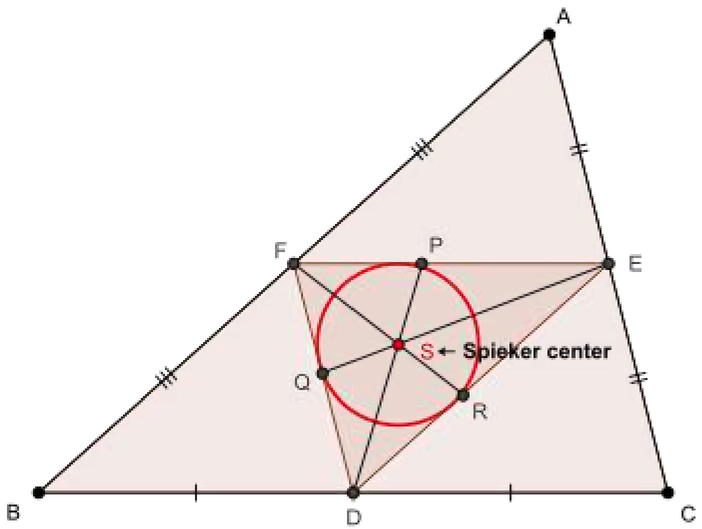
$$\therefore \sum_{cyc} h_a^2\cot\frac{A}{2} \ge \frac{2r}{R}\sum_{cyc} r_a^2\cot\frac{A}{2}\ \forall\,\Delta ABC,\text{''}=\text{'' iff }\Delta ABC\text{ is equilateral (QED)}$$

1598. In any Δ ABC with $p_a, p_b, p_c \to$
Spieker cevians, the following relationship holds :

$$p_a + p_b + p_c \le \frac{14R-r}{3}$$

Proposed by Mohamed Amine Ben Ajiba

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again}, \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\text{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\overset{(\text{ii})}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now}, [BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{Now}, b^3+c^3-abc+a\big(4m_a^2\big)=b^3+c^3+a^3-abc+a\big(2b^2+2c^2-a^2\big)-a^3$$

$$=\sum_{\text{cyc}}a^3-abc+2a.2bc\cos A=2s\big(s^2-6Rr-3r^2\big)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a\big(4m_a^2\big)\overset{(\blacksquare\blacksquare)}{=}2s\big(s^2-8Rr-3r^2+8Rr\cos A\big)$$

$$\text{We have}: \prod_{\text{cyc}}(2s+a)=8s^3+4s^2\sum_{\text{cyc}}a+2s\sum_{\text{cyc}}ab+4Rrs$$

$$=8s^3+4s^2.2s+2s\big(s^2+4Rr+r^2\big)+4Rrs$$

$$\Rightarrow\prod_{\text{cyc}}(2s+a)\overset{(\blacksquare\blacksquare\blacksquare)}{=}2s\big(9s^2+6Rr+r^2\big)\text{ and}$$

$$\sum_{\text{cyc}}(2s+b)(2s+c)=\sum_{\text{cyc}}\big(4s^2+2s(2s-a)+bc\big)$$

$$=24s^2-2s(2s)+s^2+4Rr+r^2\Rightarrow\sum_{\text{cyc}}(2s+b)(2s+c)\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}21s^2+4Rr+r^2$$

$$(\blacksquare),(\blacksquare\blacksquare)\Rightarrow p_a=\frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}\overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=}$$

$$\frac{2s}{2s(9s^2+6Rr+r^2)}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}.(2s+b)(2s+c)\text{ and analogs}$$

$$\Rightarrow p_a + p_b + p_c$$

$$= \frac{1}{9s^2+6Rr+r^2} \cdot \sum_{cyc} \left(\sqrt{(s^2 - 8Rr - 3r^2 + 8Rr\cos A)(2s+b)(2s+c)} . \sqrt{(2s+b)(2s+c)} \right)$$

$$\overset{CBS}{\leq} \frac{1}{9s^2+6Rr+r^2} \cdot \sqrt{\sum_{cyc} (s^2 - 8Rr - 3r^2 + 8Rr\cos A)(2s+b)(2s+c)} . \sqrt{\sum_{cyc} (2s+b)(2s+c)}$$

$$\overset{via\,(\blacksquare\blacksquare\blacksquare\blacksquare)}{=} \frac{1}{9s^2+6Rr+r^2} \cdot \sqrt{\begin{matrix}(s^2 - 8Rr - 3r^2)(21s^2+4Rr+r^2) + \\ 8Rr \sum_{cyc} \left((8s^2 - 2sa + bc)\cos A \right)\end{matrix}} . \sqrt{21s^2+4Rr+r^2}$$

$$= \frac{1}{9s^2+6Rr+r^2} \cdot \sqrt{\begin{matrix}(s^2 - 8Rr - 3r^2)(21s^2+4Rr+r^2) \\ +8Rr\left(8s^2.\frac{R+r}{R} - 2s.\frac{2rs}{R} + \sum_{cyc} \left(bc.\frac{b^2+c^2-a^2}{2bc} \right) \right)\end{matrix}} . \sqrt{21s^2+4Rr+r^2}$$

$$\left(\because \sum_{cyc} a\cos A = \frac{2rs}{R} \right)$$

$$= \frac{1}{9s^2+6Rr+r^2} \cdot \sqrt{\begin{matrix}(s^2 - 8Rr - 3r^2)(21s^2+4Rr+r^2) \\ +8r\left(8(R+r)s^2 - 4rs^2 + R(s^2 - 4Rr - r^2)\right)\end{matrix}} . \sqrt{21s^2+4Rr+r^2}$$

$$\Rightarrow (p_a + p_b + p_c)^2 \leq$$

$$\frac{(21s^2+4Rr+r^2)\left(21s^4 - (92Rr+30r^2)s^2 - r^2(64R^2+28Rr+3r^2)\right)}{(9s^2+6Rr+r^2)^2} \overset{?}{\leq} \frac{(14R-r)^2}{9}$$

$$\Leftrightarrow \boxed{\begin{matrix} 3969s^6 - (15876R^2 + 14364Rr + 5562)s^4 \\ -rs^2(21168R^3 + 15912R^2r + 6804Rr^2 + 855r^3) \\ -r^2(7056R^4 + 3648R^3r + 1480R^2r^2 + 344Rr^3 + 28r^4) \underset{(\bullet)}{\overset{?}{\leq}} 0 \end{matrix}}$$

Now, Rouche $\Rightarrow s^2 - (m-n) \geq 0$ and $s^2 - (m+n) \leq 0$, where

$m = 2R^2 + 10Rr - r^2$ and $n = 2(R-2r).\sqrt{R^2 - 2Rr}$

$$\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \leq 0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \overset{(\blacklozenge)}{\leq} 0$$

$\therefore 3969s^2\left(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R+r)^3\right) \leq 0 \Rightarrow$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \leq$ $3969s^2\left(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R+r)^3\right)$

$$\Leftrightarrow \boxed{\begin{matrix} (16254R - 3375r)s^4 - s^2(68796R^3 + 51606R^2r + 13608Rr^2 + 1206r^2) \\ -r(1764R^4 + 912R^3r + 370R^2r^2 + 86Rr^3 + 7r^4) \underset{(\bullet\bullet)}{\overset{?}{\leq}} 0 \end{matrix}} \text{ and}$$

$$\because (16254R - 3375r)\left(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R+r)^3\right) \overset{via\,(\blacklozenge)}{\leq} 0$$

$\therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : LHS of $(\bullet\bullet) \leq$ $(16254R - 3375r)\left(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R+r)^3\right)$

$$\Leftrightarrow \boxed{\begin{array}{c}(1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3)s^2 \\ +r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4) \overset{(\bullet\bullet\bullet)}{\geq} 0\end{array}}$$

$\boxed{Case\ 1}$ $1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3 \geq 0$ *and then* : LHS of $(\bullet\bullet\bullet)$

$$\geq r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4) > 0$$

$$\Rightarrow (\bullet\bullet\bullet) \text{ is true}$$

$\boxed{Case\ 2}$ $1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3 < 0$ *and then* : LHS of $(\bullet\bullet\bullet)$

$$= -\left(-(1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3)\right)s^2$$

$$+r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4)$$

$$\overset{Gerretsen}{\geq} -\left(-(1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3)\right)(4R^2 + 4Rr + 3r^2)$$

$$+r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4) \overset{?}{\geq} 0$$

$$\Leftrightarrow 3780t^5 + 1287t^4 - 8439t^3 - 85538t^2 + 65924t - 3704 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(3780t^4 + 11871t^3 + 3713t^2 + 17782t(t-2) + 2500) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{Euler}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \text{ is true} \therefore \text{combining both cases}, (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet)$$

$$\text{is true } \forall \Delta ABC \Rightarrow (p_a + p_b + p_c)^2 \leq \frac{(14R-r)^2}{9}$$

$$\therefore p_a + p_b + p_c \leq \frac{14R-r}{3}\ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1599. In *any* Δ ABC, the *following* relationship holds :

$$\sum_{cyc} \frac{m_a^{k+1}}{m_b + m_c - m_a} \geq \sum_{cyc} m_a^k \text{ for all } k \in \mathbb{N}$$

Proposed by Mihaly Bencze, Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall prove} : \frac{1}{2}\sum_{cyc} \frac{a^{k+1}}{s-a} \geq \sum_{cyc} a^k \text{ for all } k \in \mathbb{N}$$

$\boxed{Case\ 1}$ $k = 0$ *and then* : $\frac{1}{6}\sum_{cyc} \frac{a^{k+1}}{s-a} = \frac{1}{6}\sum_{cyc} \frac{a}{s-a} = \frac{1}{6}\sum_{cyc} \frac{a-s+s}{s-a}$

$$= \frac{1}{6}\left(-3 + \frac{s}{r^2s}.\sum_{cyc}(s-b)(s-c)\right) = \frac{1}{6}\left(-3 + \frac{(4Rr+r^2)}{r^2}\right) = \frac{2R-r}{3r} \overset{Euler}{\geq} \frac{4r-r}{3r}$$

$$\Rightarrow \frac{1}{6}\sum_{cyc} \frac{a}{s-a} \overset{(*)}{\geq} 1 \Rightarrow \frac{1}{2}\sum_{cyc} \frac{a^{k+1}}{s-a} \geq 3 = \sum_{cyc} a^k \text{ for } k = 0$$

$\boxed{Case\ 2}$ $k \in \mathbb{N}^*$ *and then* : $\frac{1}{2}\sum_{cyc} \frac{a^{k+1}}{s-a} \overset{Chebyshev}{\geq} \frac{1}{6}\left(\sum_{cyc} a^k\right)\left(\sum_{cyc} \frac{a}{s-a}\right)$

$$\left(\because \text{WLOG assuming } a \ge b \ge c \Rightarrow a^k \ge b^k \ge c^k \text{ and } \frac{a}{s-a} \ge \frac{b}{s-b} \ge \frac{c}{s-c}\right)$$
$$\text{as } k > 0 \text{ since } k \in \mathbb{N}^*$$
$$\overset{\text{via } (*)}{\ge} \sum_{cyc} a^k \text{ for all } k \in \mathbb{N}^* \therefore \text{for all } k \in \mathbb{N}, \frac{1}{2}\sum_{cyc}\frac{a^{k+1}}{s-a} \ge \sum_{cyc} a^k$$
$$\Rightarrow \sum_{cyc}\frac{a^{k+1}}{b+c-a} \overset{(**)}{\ge} \sum_{cyc} a^k \text{ and implementing } (**) \text{ on } a \text{ triangle with sides}$$
$$m_a, m_b, m_c, \text{we arrive at}: \sum_{cyc}\frac{m_a^{k+1}}{m_b+m_c-m_a} \ge \sum_{cyc} m_a^k$$
$$\text{for all } k \in \mathbb{N} \text{ and } \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1600.

In any acute Δ ABC, the following relationship holds :

$$h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)+h_b^2\left(\frac{1}{m_c^2}+\frac{1}{m_a^2}\right)+h_c^2\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right) \ge 6$$

Proposed by Lam Tran-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} b^2c^2\sec^4\frac{A}{2} = \left(\sum_{cyc} bc\sec^2\frac{A}{2}\right)^2 - 2\sum_{cyc} bc.ca.\sec^2\frac{A}{2}\sec^2\frac{B}{2}$$
$$= \left(\sum_{cyc} bc + 4Rrs.\sum_{cyc}\frac{\tan^2\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\right)^2 - 8Rrs.\frac{16R^2}{s^2}.\sum_{cyc} a\cos^2\frac{A}{2}$$
$$= \left(s^2+4Rr+r^2+r\sum_{cyc}\left(\left(1+\tan^2\frac{A}{2}\right)r_a\right)\right)^2 - \frac{64R^3r}{s}\sum_{cyc} a(1+\cos A)$$
$$= \left(s^2+4Rr+r^2+r(4R+r)+\frac{r}{s^2}\left((4R+r)^3-3.64R^3.\frac{s^2}{16R^2}\right)\right)^2$$
$$-\frac{64R^3r}{s}\left(2s+\frac{2rs}{R}\right)\left(\because r_b+r_c = 4R\cos^2\frac{A}{2} \text{ and analogs}\right)$$
$$\Rightarrow \sum_{cyc} b^2c^2\sec^4\frac{A}{2} =$$
$$\frac{\left(s^2(s^2+4Rr+r^2)+r(4R+r)s^2+r(4R+r)^3-12Rrs^2\right)^2-128R^2r(R+r)s^4}{s^4}$$
$$\to (1)$$

$$\text{Also, } \sum_{cyc} \sec^4\frac{A}{2} = \left(\sum_{cyc}\sec^2\frac{A}{2}\right)^2 - 2\sum_{cyc}\sec^2\frac{A}{2}\sec^2\frac{B}{2}$$

$$= \frac{\left(s^2+(4R+r)^2\right)^2}{s^4} - 2.\frac{16R^2}{s^2}.\frac{4R+r}{2R}$$

$$\Rightarrow \sum_{cyc}\sec^4\frac{A}{2} = \frac{\left(s^2+(4R+r)^2\right)^2 - 16R(4R+r)s^2}{s^4} \rightarrow (2)$$

$$\text{Now, } \because m_a \le 2R\cos^2\frac{A}{2} \text{ and analogs } \forall \text{ acute } \Delta\, ABC, \therefore \sum_{cyc} h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right)$$

$$\ge \sum_{cyc}\left(\frac{b^2c^2}{4R^2}\left(\frac{1}{4R^2}\left(\sum_{cyc}\sec^4\frac{A}{2} - \sec^4\frac{A}{2}\right)\right)\right)$$

$$= \frac{\left(\sum_{cyc}\sec^4\frac{A}{2}\right)\left(\sum_{cyc}b^2c^2\right) - \sum_{cyc}b^2c^2\sec^4\frac{A}{2}}{16R^4}$$

$$\overset{\text{via (1) and (2)}}{=} \frac{\left(\left(s^2+(4R+r)^2\right)^2 - 16R(4R+r)s^2\right)\left(\left(s^2+4Rr+r^2\right)^2 - 16Rrs^2\right)}{16R^4s^4}$$

$$- \frac{\left(s^2\left(s^2+4Rr+r^2\right) + r(4R+r)s^2 + r(4R+r)^3 - 12Rrs^2\right)^2 - 128R^2r(R+r)s^4}{16R^4s^4}$$

$$= \frac{32R^2s^2\left(-s^4+\left(8R^2+16Rr+2r^2\right)s^2 - r(4R+r)^3\right)}{16R^4s^4} \overset{?}{\ge} 6$$

$$\Leftrightarrow \boxed{s^4 - \left(5R^2+16Rr+2r^2\right)s^2 + r(4R+r)^3 \underset{(*)}{\overset{?}{\le}} 0}$$

Now, via Wallker and Gerretsen, $\forall$ acute $\Delta\, ABC$,
$\left(s^2-2R^2-8Rr-3r^2\right)\left(s^2-4R^2-4Rr-3r^2\right) \le 0$ and so,
in order to prove $(*)$, it suffices to prove :
LHS of $(*) \le \left(s^2-2R^2-8Rr-3r^2\right)\left(s^2-4R^2-4Rr-3r^2\right)$

$$\Leftrightarrow \boxed{\left(R^2-4Rr+4r^2\right)s^2 \overset{(**)}{\le} 8R^4 - 24R^3r + 2R^2r^2 + 24Rr^3 + 8r^4}$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\le} \left(R^2-4Rr+4r^2\right)\left(4R^2+4Rr+3r^2\right) \overset{?}{\le}$$

$$8R^4 - 24R^3r + 2R^2r^2 + 24Rr^3 + 8r^4 \Leftrightarrow 4t^4 - 12t^3 - t^2 + 20t - 4 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left(4t^2+4t-1\right) \overset{?}{\ge} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore h_a^2\left(\frac{1}{m_b^2}+\frac{1}{m_c^2}\right) + h_b^2\left(\frac{1}{m_c^2}+\frac{1}{m_a^2}\right) + h_c^2\left(\frac{1}{m_a^2}+\frac{1}{m_b^2}\right) \ge 6$$

$\forall\, \Delta\, ABC$, " $=$ " iff $\Delta\, ABC$ is equilateral (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

We will first prove that, for any acute triangle ABC, we have

$$m_a \le \sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2}$$

Since $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{(b+c)^2-a^2}{4bc}}$**, then we have**

$$\left(\sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2}\right)^2 - {m_a}^2 = \frac{(b^2+c^2)\left[(b+c)^2-a^2\right]}{8bc} - \frac{2(b^2+c^2)-a^2}{4}$$

$$= \frac{(b^2+c^2-a^2)(b-c)^2}{8bc} \ge 0,$$

which is true for the acute triangle ABC.

Using this result, and since $h_a = \frac{bc}{2R}$ **(and analogs) and** $a = 4R\,sin\frac{A}{2}cos\frac{A}{2}$**, we have**

$$m_a \le \sqrt{\frac{b^2+c^2}{2}}.\cos\frac{A}{2} = \sqrt{\frac{b^2+c^2}{2}}.\frac{a}{4R\,sin\frac{A}{2}} = \sqrt{\frac{{h_c}^2+{h_b}^2}{8}}\csc\frac{A}{2} \text{ (and analogs)}$$

Therefore

$$\sum_{cyc} {h_a}^2\left(\frac{1}{{m_b}^2}+\frac{1}{{m_c}^2}\right) = \sum_{cyc}\frac{{h_b}^2+{h_c}^2}{{m_a}^2} \ge \sum_{cyc} 8\sin^2\frac{A}{2} = 4\sum_{cyc}(1-\cos A) \overset{Jensen}{\ge} 4\left(3-\frac{3}{2}\right) = 6.$$

Equality holds iff ΔABC **is equilateral.**

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru