# ROMANIAN MATHEMATICAL MAGAZINE

**Prove that:**

$$\sum_{n=1}^{\infty}\frac{6^nH_n^{(2)}+3^nnH_n^{(2)}+2^nn^2H_n^{(3)}}{(-1)^nn6^n}=\frac{\pi^2}{12}\ln(2)-\zeta(3)+\frac{2}{3}Li_2\left(-\frac{1}{2}\right)$$
$$+\frac{3}{4}Li_2\left(-\frac{1}{3}\right)-\frac{3}{16}Li_3\left(-\frac{1}{3}\right)$$

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$$\sigma=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(2)}}{n}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(2)}}{2^n}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(3)}}{3^n}n=\sigma_1+\sigma_2+\sigma_3$$

$Notes:$

$$\left\{\sum_{n=1}^{\infty}(-1)^nH_n^{(q)}x^n=\frac{Li_q(-x)}{1+x}\right\};\left\{\frac{d}{dx}\sum_{n=1}^{\infty}(-1)^n\,H_n^{(q)}x^n=\frac{d}{dx}\frac{Li_q(-x)}{1+x}\right\}$$

$$\sigma_1=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^{(2)}}{n}=\sum_{n=1}^{\infty}(-1)^nH_n^{(2)}\int_0^1x^{n-1}dx=\int_0^1\frac{1}{x}\sum_{n=1}^{\infty}(-1)^nH_n^{(2)}x^ndx=\int_0^1\frac{Li_2(-x)}{x(1+x)}dx$$
$$=$$

$$=\int_0^1\frac{Li_2(-x)}{x}dx-\int_0^1\frac{Li_2(-x)}{1+x}dx$$
$$\dot{I}BP$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1x^{n-1}dx-[ln(1+x)Li_2(-x)]_0^1-\int_0^1\frac{\frac{ln^2(1+x)}{x}}{\{\frac{1}{1+x}=x\}}dx=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-Li_2(-1)ln(2)-\int_{\frac{1}{2}}^1\frac{ln^2(x)}{x}dx-\int_{\frac{1}{2}}^1\frac{ln^2(x)}{1-x}dx$$

$$=-\eta(3)+\frac{\zeta(2)ln(2)}{2}-[\frac{ln^3(x)}{3}]_{\frac{1}{2}}^1-\int_0^1\frac{ln^2(x)}{1-x}dx+\int_0^{\frac{1}{2}}\frac{ln^2(x)}{1-x}dx=$$

$$=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)-\frac{ln^3\left(\frac{1}{2}\right)}{3}-\sum_{n=0}^{\infty}\int_0^1x^nln^2(x)dx+\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}}x^nln^2(x)dx=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)$$
$$+\frac{ln^3(2)}{3}-2\sum_{n=0}^{\infty}\frac{1}{(n+1)^3}+\sum_{n=0}^{\infty}([\frac{x^{n+1}ln^2(x)}{n+1}]_0^{\frac{1}{2}}$$
$$-\frac{2}{n+1}\int_0^{\frac{1}{2}}x^nln(x)dx=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)+\frac{ln^3(2)}{3}-2\zeta(3)-\frac{ln^3(2)}{3}+\frac{7\zeta(3)}{4}$$
$$=\frac{\pi^2\ln(2)}{12}-\frac{3}{4}\zeta(3)-\frac{\zeta(3)}{4}=\frac{\pi^2\ln(2)}{12}-\zeta(3)$$

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$$\sigma_2=\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{2^n}=\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n H_n^{(2)}=\frac{Li_2\left(-\frac{1}{2}\right)}{1+\frac{1}{2}}=\frac{2Li_2\left(-\frac{1}{2}\right)}{3}$$

$$\sigma_3=\sum_{n=1}^{\infty}\frac{(-1)^n n H_n^{(3)}}{3^n}$$

$$\left\{\frac{d}{dx}\sum_{n=1}^{\infty}(-1)^n x^n H_n^{(3)}=\frac{d}{dx}\frac{Li_3(-x)}{1+x};\ \sum_{n=1}^{\infty}(-1)^n n x^{n-1}H_n^{(3)}=\frac{(x+1)Li_2(-x)-xLi_3(-x)}{x(x+1)^2}\right\}$$

$$\sigma_3=\sum_{n=1}^{\infty}(-1)^n n\left(\frac{1}{3}\right)^n H_n^{(3)}=\frac{\left(1+\frac{1}{3}\right)Li_2\left(-\frac{1}{3}\right)-\frac{1}{3}Li_3\left(-\frac{1}{3}\right)}{\left(1+\frac{1}{3}\right)^2}=\frac{3}{4}Li_2\left(-\frac{1}{3}\right)-\frac{3}{16}Li_3\left(-\frac{1}{3}\right)$$

$$\sum_{n=1}^{\infty}\frac{6^n H_n^{(2)}+3^n n H_n^{(2)}+2^n n^2 H_n^{(3)}}{(-1)^n 6^n n}=\sigma_1+\sigma_2+\sigma_3=\frac{\pi^2 ln(2)}{12}-\zeta(3)+\frac{2}{3}Li_2\left(-\frac{1}{2}\right)$$

$$+\frac{3}{4}Li_2\left(-\frac{1}{3}\right)-\frac{3}{16}Li_3\left(-\frac{1}{3}\right)$$