ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\sum_{n=1}^{\infty}\frac{H_n - H_{\frac{n}{2}}}{n}\binom{2n}{n}\left(-\frac{1}{4}\right)^n$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution 1 by proposer

$$\sum_{n=1}^{\infty}\frac{H_n - H_{\frac{n}{2}}}{n}\binom{2n}{n}\left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty}\frac{\Gamma(2n+1)}{n!\,n2^{2n}\Gamma(n+1)}(-1)^n\int_0^1\left(\frac{1-x^n}{1-x}-\frac{1-x^{\frac{n}{2}}}{1-x}\right)dx$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\,n}(-1)^n\int_0^1\frac{x^{\frac{n}{2}}-x^n}{1-x}dx$$

$$=\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(-1)^n\int_0^1\left(\frac{1}{2}x^{\frac{n}{2}-1}-x^{n-1}\right)\ln(1-x)\,dx$$

$$=\int_0^1\left(\frac{1}{2}\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(-1)^n x^{\frac{n}{2}-1}-\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}(-1)^n x^{n-1}\right)\ln(1-x)\,dx$$

$$=\int_0^1\left(\frac{1}{2}\frac{1}{x\sqrt{1+\sqrt{x}}}-\frac{1}{x\sqrt{1+x}}+\frac{1}{2x}\right)\ln(1-x)\,dx$$

$$=\frac{1}{2}\int_0^1\frac{1}{x\sqrt{1+\sqrt{x}}}\ln(1-x)\,dx-\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1-x)\,dx-\frac{\pi^2}{12}=$$

$$=\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1-x)\,dx-\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1-x)\,dx-\frac{\pi^2}{12}=$$

$$=\int_0^1\frac{1}{x\sqrt{1+x}}\ln(1+x)\,dx-\frac{\pi^2}{12}$$

$$=4\int_1^{\sqrt{2}}\frac{1}{(u^2-1)}\ln u\,du-\frac{\pi^2}{12}=-4\int_{\frac{1}{\sqrt{2}}}^1\frac{1}{1-t^2}\ln t\,dt-\frac{\pi^2}{12}=$$

$$=-2\int_0^{\frac{\sqrt{2}-1}{\sqrt{2}+1}}\frac{1}{x}\ln\left(\frac{1-x}{1+x}\right)dx-\frac{\pi^2}{12}=2Li_2\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)-2Li_2\left(-\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)-\frac{\pi^2}{12}$$

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Solution 2 by Bui Hong Suc-Vietnam

By Taylor series:

$$\frac{1}{\sqrt{1+x}}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{4^n}(-x)^n=1+\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{4^n}(-x)^n\to\sum_{n=1}^{\infty}\binom{2n}{n}\left(-\frac{1}{4}\right)^n x^n=\frac{1-\sqrt{1+x}}{\sqrt{1+x}}$$

Multiply through by $\frac{\ln(1+x)}{x}$ then integrate using $\int_0^1 x^{n-1}\ln(1+x)\,dx=\frac{H_n-H_{\frac{n}{2}}}{n}$,

$$\therefore\sum_{n=1}^{\infty}\binom{2n}{n}\left(-\frac{1}{4}\right)^n\frac{H_n-H_{\frac{n}{2}}}{n}=\int_0^1\frac{\left(1-\sqrt{1+x}\right)\ln(1+x)}{x\sqrt{1+x}}dx=$$

$$=\int_0^1\frac{\left(1-\sqrt{1+x}\right)\left(1+\sqrt{1+x}\right)\ln(1+x)}{x\left(1+\sqrt{1+x}\right)\sqrt{1+x}}dx$$

$$=2\int_0^1\frac{\ln\left(\frac{1}{\sqrt{1+x}}\right)}{\left(1+\sqrt{1+x}\right)\sqrt{1+x}}dx\overset{\frac{1}{v^2}=1+x}{=}4\int_{\frac{1}{\sqrt{2}}}^{1}\frac{\ln(v)}{\left(1+\frac{1}{b}\right)\frac{1}{v}}\frac{dv}{v^3}=$$

$$=4\int_{\frac{1}{\sqrt{2}}}^{1}\frac{\ln(v)}{(v+1)v}=4\int_{\frac{1}{\sqrt{2}}}^{1}\frac{\ln(v)}{v}dv-4\int_{\frac{1}{\sqrt{2}}}^{1}\frac{\ln(v)}{v+1}dv$$

$$=2\ln^2(v)\Big|_{\frac{1}{\sqrt{2}}}^{1}-4\ln(v)\ln(1+v)\Big|_{\frac{1}{\sqrt{2}}}^{1}+4\int_{\frac{1}{\sqrt{2}}}^{1}\frac{\ln\left(1-(-v)\right)}{-v}d(-v)=$$

$$=-2\ln^2\sqrt{2}-4\ln\left(\sqrt{2}\right)\ln\left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)-4Li_2(-v)\Big|_{\frac{1}{\sqrt{2}}}^{1}$$

$$=4Li_2\left(-\frac{1}{\sqrt{2}}\right)-4Li_2(-1)+2\ln^2\sqrt{2}-4\ln\left(\sqrt{2}\right)\ln\left(1+\sqrt{2}\right)=$$

$$=4Li_2\left(-\frac{1}{\sqrt{2}}\right)+\frac{\pi^2}{3}+2\ln^2\sqrt{2}-2\ln(2)\ln\left(1+\sqrt{2}\right)$$

Hence:

$$\sum_{n=1}^{\infty}\binom{2n}{n}\left(-\frac{1}{4}\right)^n\frac{H_n-H_{\frac{n}{2}}}{n}=4Li_2\left(-\frac{1}{\sqrt{2}}\right)+\frac{\pi^2}{3}+2\ln^2\sqrt{2}-2\ln(2)\ln\left(1+\sqrt{2}\right)$$