ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega=\sum_{k=1}^{\infty}\frac{H_k 2^{-k}}{(k+1)}$$

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Solution by Shirvan Tahirov-Azerbaijan

$$\sum_{k=1}^{\infty}\frac{H_k 2^{-k}}{(k+1)}=\sum_{k=1}^{\infty}\frac{H_k}{2^k(k+1)}=\sum_{k=1}^{\infty}\left(\frac{H_k}{2^k}\cdot\frac{1}{(k+1)}\right)=\sum_{k=1}^{\infty}\frac{H_k}{2^k}\int_0^1 x^k dx=$$

$$\int_0^1\left(\sum_{k=1}^{\infty}H_k\left(\frac{x}{2}\right)^k\right)dx=-\int_0^1\frac{\ln\left(1-\frac{x}{2}\right)}{\left(1-\frac{x}{2}\right)}dx=-\left(-\ln\left(1-\frac{x}{2}\right)\right)\Big|_0^1=$$

$$=-\left(-ln^2(2)\right)=ln^2(2)$$

Note :

$$\sum_{k=1}^{\infty}H_k x^k=-\frac{\ln(1-x)}{1-x}$$

Where H_k is the harmonic number.