ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\sum_{n=1}^{\infty}\frac{1}{n2^{2n}}\binom{2n}{n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=\frac{\pi^2}{12}$$

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$$\sigma=\sum_{n=1}^{\infty}\frac{1}{n2^{2n}}\binom{2n}{n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k};$$

Notes:

$$\left\{\text{Skew harmonic series: } H_n^-=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k},\ H_{2n}^-=\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=\int_0^1\frac{1-x^{2n}}{1+x}dx\right.$$

$$\left.=H_{2n}-H_n\right\}$$

$$\left\{\text{Generating Binomial series: }\sum_{n=1}^{\infty}\frac{x^{2n-1}}{4^n}\binom{2n}{n}=\frac{1}{x\sqrt{1-x^2}}-\frac{1}{x};\int\sum_{n=1}^{\infty}\frac{x^{2n-1}}{4^n}\binom{2n}{n}dx\right.$$

$$\left.=\int\frac{1-\sqrt{1-x^2}}{x\sqrt{1-x^2}}dx\right\}$$

$$\sum_{n=1}^{\infty}\frac{x^{2n}}{n4^n}\binom{2n}{n}=-2\ln(x)-2\tanh^{-1}\sqrt{1-x^2}$$

$$\sigma=\sum_{n=1}^{\infty}\frac{H_{2n}^-}{n4^n}\binom{2n}{n}=\sum_{n=1}^{\infty}\frac{H_{2n}-H_n}{n4^n}\binom{2n}{n}=\sum_{n=1}^{\infty}\frac{1}{n4^n}\binom{2n}{n}\int_0^1\frac{1-x^{2n}}{1+x}dx=$$

$$=\int_0^1\frac{1}{1+x}\left(\sum_{n=1}^{\infty}\frac{1}{n4^n}\binom{2n}{n}-\sum_{n=1}^{\infty}\frac{x^{2n}}{n4^n}\binom{2n}{n}\right)dx$$

$$=\int_0^1\frac{-2\ln(1)-2\tanh^{-1}(0)+2\tanh^{-1}\sqrt{1-x^2}+2\ln(x)}{1+x}dx=$$

$$=2\int_0^1\frac{\tanh^{-1}\sqrt{1-x^2}}{1+x}dx-2\int_0^1\frac{\ln(x)}{1+x}dx\quad \text{Using İBP method: }\{v=\ln(1+x),u$$

$$=\tanh^{-1}\sqrt{1-x^2},\frac{du}{dx}=-\frac{1}{x\sqrt{1-x^2}}\}$$

$$\sigma=2[\ln(1+x)\tanh^{-1}\sqrt{1-x^2}]_0^1+2\int_0^1\frac{\ln(1+x)}{x\sqrt{1-x^2}}dx-\frac{\pi^2}{6}=2\int_0^1\frac{\ln(1+x)}{x\sqrt{1-x^2}}dx-\frac{\pi^2}{6}$$

Using Feynman's integration method

$$I(a)=2\int_0^1\frac{\ln(1+ax)}{x\sqrt{1-x^2}}dx,\frac{d}{da}I(a)=2\int_0^1\frac{1}{(1+ax)\sqrt{1-x^2}}dx$$

Subtitation $\{x=\sin(t), dx=\cos(t)\,dt\}$

$$I'(a)=2\int_0^{\frac{\pi}{2}}\frac{1}{a\sin(t)+1}dt,\left\{\tan\left(\frac{t}{2}\right)=u,dt=\frac{2du}{1+u^2},\sin(t)=\frac{2u}{1+u^2},u[1;0]\right\}$$

$$I'(a)=4\int_0^1\frac{1}{\frac{2au}{1+u^2}+1}\frac{du}{1+u^2}=4\int_0^1\frac{1}{u^2+2ua+1}du=4\left[\frac{\tan^{-1}\left(\frac{u+a}{\sqrt{1-a^2}}\right)}{\sqrt{1-a^2}}\right]_0^1=$$

$$=\frac{4}{\sqrt{1-a^2}}\left(\tan^{-1}\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right)\right);\ \{I(1)=\sigma\quad I(0)=0\}$$

$$I=4\int_0^1\frac{1}{\sqrt{1-a^2}}\left(\tan^{-1}\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right)\right)da$$

$$Subtitation:\left\{\tan^{-1}\left(\frac{a+1}{\sqrt{1-a^2}}\right)-tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right);\ d\theta=-\frac{1}{2\sqrt{1-a^2}};\theta\left[0;\frac{\pi}{4}\right]\right\}$$

$$I=8\int_0^{\frac{\pi}{4}}\theta d\theta=8\left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{4}}=\frac{8\pi^2}{32}=\frac{\pi^2}{4}$$

$$\sigma=\sum_{n=1}^{\infty}\frac{1}{n2^{2n}}\binom{2n}{n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=\frac{\pi^2}{4}-\frac{\pi^2}{6}=\frac{\pi^2}{12}$$